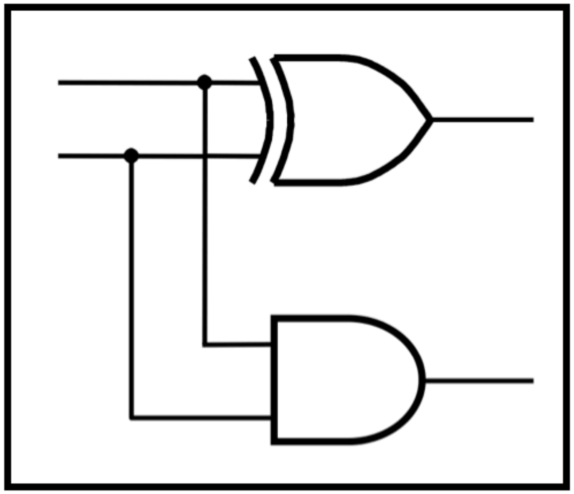


CprE 281: Digital Logic

Instructor: Alexander Stoytchev

http://www.ece.iastate.edu/~alexs/classes/

Minimization

CprE 281: Digital Logic
Iowa State University, Ames, IA

Administrative Stuff

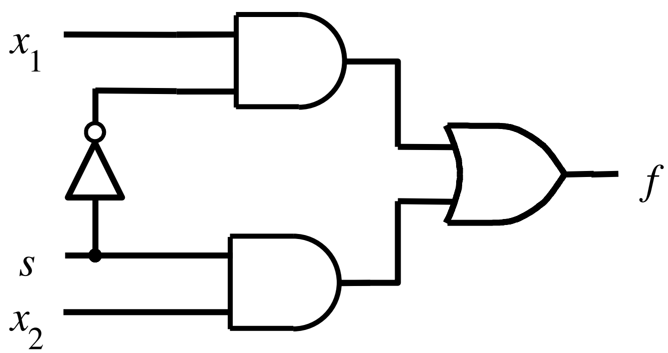
- HW4 is out
- It is due on Monday Sep 19 @ 4 pm
- It is posted on the class web page
- I also sent you an e-mail with the link.

Example:
K-Map for the 2-1 Multiplexer

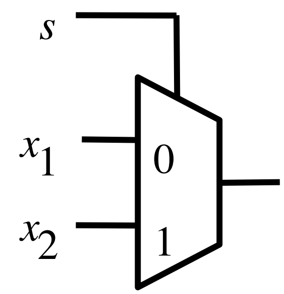
2-1 Multiplexer (Definition)

- Has two inputs: x_1 and x_2
- Also has another input line s
- If s=0, then the output is equal to x_1
- If s=1, then the output is equal to x_2

Circuit for 2-1 Multiplexer



(b) Circuit



(c) Graphical symbol

[Figure 2.33b-c from the textbook]

Truth Table for a 2-1 Multiplexer

$s\ x_1\ x_2$	$f(s, x_1, x_2)$
0 0 0	0
0 0 1	0
0 1 0	1
0 1 1	1
1 0 0	0
1 0 1	1
1 1 0	0
1 1 1	1

[Figure 2.33a from the textbook]

Let’s Draw the K-map

$s\ x_1\ x_2$	$f(s, x_1, x_2)$
0 0 0	0
0 0 1	0
0 1 0	1
0 1 1	1
1 0 0	0
1 0 1	1
1 1 0	0
1 1 1	1

[Figure 2.33a from the textbook]

Let’s Draw the K-map

	$s\ x_1\ x_2$	$f(s, x_1, x_2)$
m_0	0 0 0	0
m_1	0 0 1	0
m_2	0 1 0	1
m_3	0 1 1	1
m_4	1 0 0	0
m_5	1 0 1	1
m_6	1 1 0	0
m_7	1 1 1	1

Let's Draw the K-map

	$s\ x_1\ x_2$	$f(s, x_1, x_2)$
m_0	0 0 0	0
m_1	0 0 1	0
m_2	0 1 0	1
m_3	0 1 1	1
m_4	1 0 0	0
m_5	1 0 1	1
m_6	1 1 0	0
m_7	1 1 1	1

$s \backslash x_1x_2$	00	01	11	10
0	m_0	m_2	m_6	m_4
1	m_1	m_3	m_7	m_5

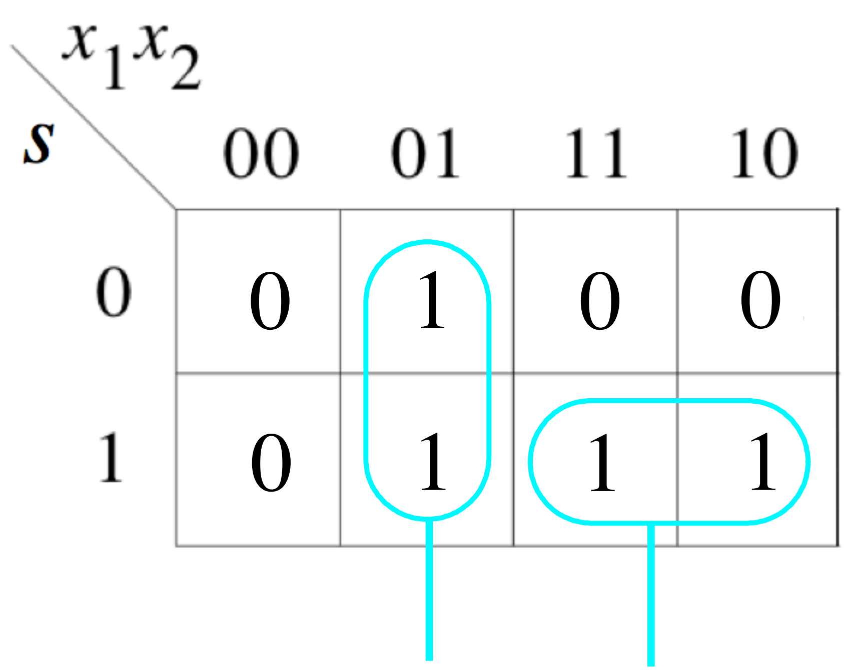
Let's Draw the K-map

	$s\ x_1\ x_2$	$f(s, x_1, x_2)$
m_0	0 0 0	0
m_1	0 0 1	0
m_2	0 1 0	1
m_3	0 1 1	1
m_4	1 0 0	0
m_5	1 0 1	1
m_6	1 1 0	0
m_7	1 1 1	1

s \ $x_1 x_2$	00	01	11	10
0	0	1	0	0
1	0	1	1	1

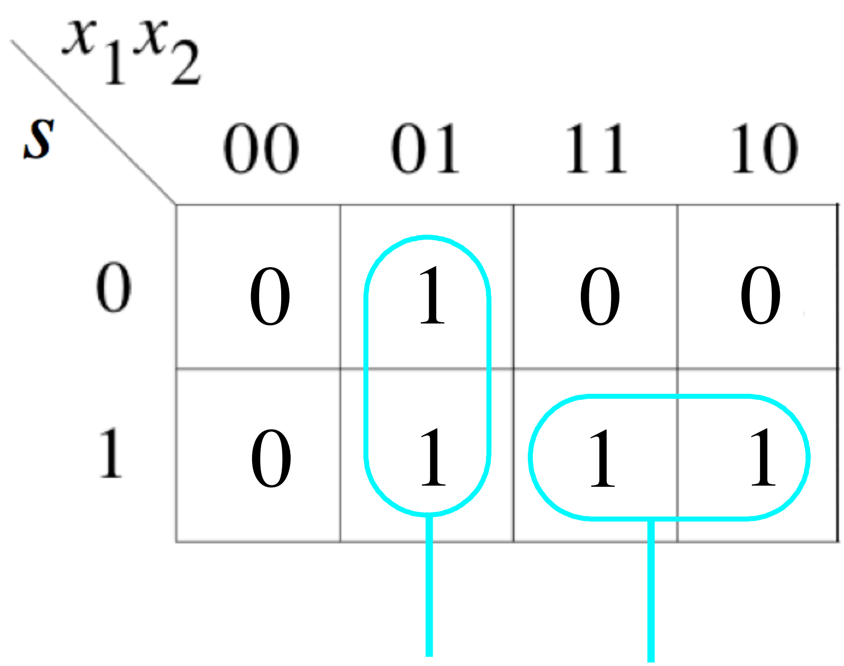
Let's Draw the K-map

	$s\ x_1\ x_2$	$f(s, x_1, x_2)$
m_0	0 0 0	0
m_1	0 0 1	0
m_2	0 1 0	1
m_3	0 1 1	1
m_4	1 0 0	0
m_5	1 0 1	1
m_6	1 1 0	0
m_7	1 1 1	1



Let's Draw the K-map

	$s\ x_1\ x_2$	$f(s, x_1, x_2)$
m_0	0 0 0	0
m_1	0 0 1	0
m_2	0 1 0	1
m_3	0 1 1	1
m_4	1 0 0	0
m_5	1 0 1	1
m_6	1 1 0	0
m_7	1 1 1	1



$$f(s, x_1, x_2) = \overline{x}_1 x_2 + s\, x_1$$

Let's Draw the K-map

	$s\ x_1\ x_2$	$f(s, x_1, x_2)$
m_0	0 0 0	0
m_1	0 0 1	0
m_2	0 1 0	1
m_3	0 1 1	1
m_4	1 0 0	0
m_5	1 0 1	1
m_6	1 1 0	0
m_7	1 1 1	1

$s \backslash x_1 x_2$	00	01	11	10
0	0	1	0	0
1	0	1	1	1

$$f(s, x_1, x_2) = \overline{x}_1 x_2 + s x_1$$

Something is wrong!

Compare this with the SOP derivation

Let’s Derive the SOP form

$s\ x_1\ x_2$	$f(s, x_1, x_2)$
0 0 0	0
0 0 1	0
0 1 0	1
0 1 1	1
1 0 0	0
1 0 1	1
1 1 0	0
1 1 1	1

Let's Derive the SOP form

$s\ x_1\ x_2$	$f(s, x_1, x_2)$
0 0 0	0
0 0 1	0
0 1 0	1
0 1 1	1
1 0 0	0
1 0 1	1
1 1 0	0
1 1 1	1

Let's Derive the SOP form

$s\ x_1\ x_2$	$f(s, x_1, x_2)$
0 0 0	0
0 0 1	0
0 1 0	1
0 1 1	1
1 0 0	0
1 0 1	1
1 1 0	0
1 1 1	1

Where should we put the negation signs?

$s\ x_1\ x_2$

$s\ x_1\ x_2$

$s\ x_1\ x_2$

$s\ x_1\ x_2$

Let's Derive the SOP form

$s\ x_1\ x_2$	$f(s, x_1, x_2)$	
0 0 0	0	
0 0 1	0	
0 1 0	1	$\overline{s}\ x_1\ \overline{x}_2$
0 1 1	1	$\overline{s}\ x_1\ x_2$
1 0 0	0	
1 0 1	1	$s\ \overline{x}_1\ x_2$
1 1 0	0	
1 1 1	1	$s\ x_1\ x_2$

Let's Derive the SOP form

$s\ x_1\ x_2$	$f(s, x_1, x_2)$	
0 0 0	0	
0 0 1	0	
0 1 0	1	$\overline{s}\ x_1\ \overline{x}_2$
0 1 1	1	$\overline{s}\ x_1\ x_2$
1 0 0	0	
1 0 1	1	$s\ \overline{x}_1\ x_2$
1 1 0	0	
1 1 1	1	$s\ x_1\ x_2$

$$f(s, x_1, x_2) = \overline{s}\ x_1\ \overline{x}_2 + \overline{s}\ x_1\ x_2 + s\ \overline{x}_1\ x_2 + s\ x_1\ x_2$$

Let's simplify this expression

$$f(s, x_1, x_2) = \overline{s}\, x_1 \overline{x_2} + \overline{s}\, x_1 x_2 + s\, \overline{x_1}\, x_2 + s\, x_1 x_2$$

Let's simplify this expression

$$f(s, x_1, x_2) = \overline{s}\, x_1 \overline{x_2} + \overline{s}\, x_1 x_2 + s\, \overline{x_1}\, x_2 + s\, x_1 x_2$$

$$f(s, x_1, x_2) = \overline{s}\, x_1 (\overline{x_2} + x_2) + s\, (\overline{x_1} + x_1) x_2$$

Let's simplify this expression

$$f(s, x_1, x_2) = \overline{s}\, x_1\, \overline{x_2} + \overline{s}\, x_1\, x_2 + s\, \overline{x_1}\, x_2 + s\, x_1\, x_2$$

$$f(s, x_1, x_2) = \overline{s}\, x_1\, (\overline{x_2} + x_2) + s\, (\overline{x_1} + x_1)\, x_2$$

$$f(s, x_1, x_2) = \overline{s}\, x_1 + s\, x_2$$

Let's Draw the K-map again

$s\ x_1\ x_2$	$f(s, x_1, x_2)$
0 0 0	0
0 0 1	0
0 1 0	1
0 1 1	1
1 0 0	0
1 0 1	1
1 1 0	0
1 1 1	1

[Figure 2.33a from the textbook]

Let's Draw the K-map again

	$s\ x_1\ x_2$	$f(s, x_1, x_2)$
m_0	0 0 0	0
m_1	0 0 1	0
m_2	0 1 0	1
m_3	0 1 1	1
m_4	1 0 0	0
m_5	1 0 1	1
m_6	1 1 0	0
m_7	1 1 1	1

Let's Draw the K-map again

	$s\ x_1\ x_2$	$f(s, x_1, x_2)$
m_0	0 0 0	0
m_1	0 0 1	0
m_2	0 1 0	1
m_3	0 1 1	1
m_4	1 0 0	0
m_5	1 0 1	1
m_6	1 1 0	0
m_7	1 1 1	1

s \ x_1x_2	00	01	11	10
0	m_0	m_2	m_6	m_4
1	m_1	m_3	m_7	m_5

Let's Draw the K-map again

	$s\ x_1\ x_2$	$f(s, x_1, x_2)$
m_0	0 0 0	0
m_1	0 0 1	0
m_2	0 1 0	1
m_3	0 1 1	1
m_4	1 0 0	0
m_5	1 0 1	1
m_6	1 1 0	0
m_7	1 1 1	1

$s \backslash x_1x_2$	00	01	11	10
0	m_0	m_2	m_6	m_4
1	m_1	m_3	m_7	m_5

Let's Draw the K-map again

	$s\ x_1\ x_2$	$f(s, x_1, x_2)$
m_0	0 0 0	0
m_1	0 0 1	0
m_2	0 1 0	1
m_3	0 1 1	1
m_4	1 0 0	0
m_5	1 0 1	1
m_6	1 1 0	0
m_7	1 1 1	1

x_2 \ $s\ x_1$	00	01	11	10
0	m_0	m_2	m_6	m_4
1	m_1	m_3	m_7	m_5

The order of the labeling matters.

Let's Draw the K-map again

	$s\ x_1\ x_2$	$f(s, x_1, x_2)$
m_0	0 0 0	0
m_1	0 0 1	0
m_2	0 1 0	1
m_3	0 1 1	1
m_4	1 0 0	0
m_5	1 0 1	1
m_6	1 1 0	0
m_7	1 1 1	1

x_2 \ $s\ x_1$	00	01	11	10
0	m_0	m_2	m_6	m_4
1	m_1	m_3	m_7	m_5

Let's Draw the K-map again

	$s\ x_1\ x_2$	$f(s, x_1, x_2)$
m_0	0 0 0	0
m_1	0 0 1	0
m_2	0 1 0	1
m_3	0 1 1	1
m_4	1 0 0	0
m_5	1 0 1	1
m_6	1 1 0	0
m_7	1 1 1	1

x_2 \ $s\ x_1$	00	01	11	10
0	0	1	0	0
1	0	1	1	1

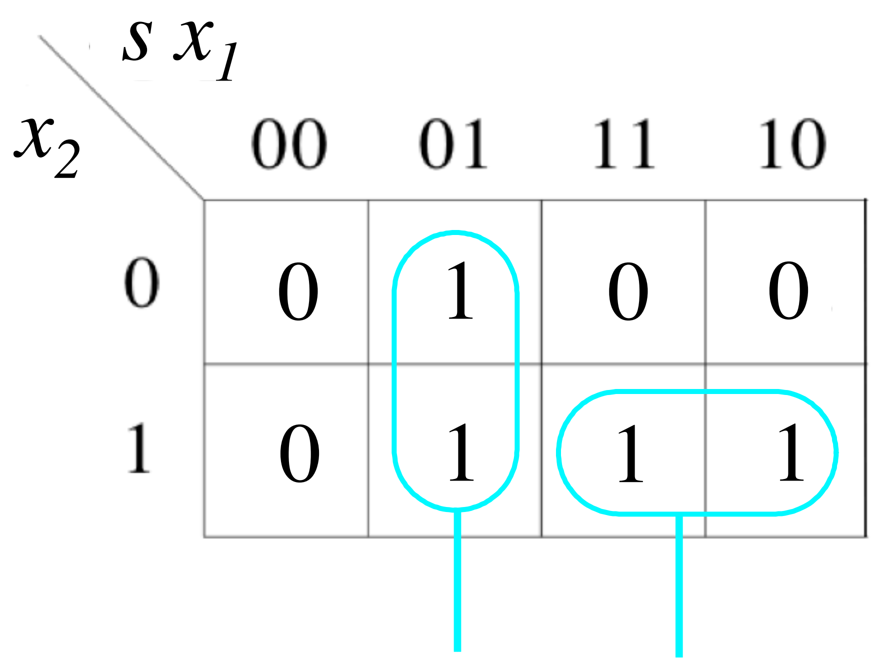
Let's Draw the K-map again

	$s\ x_1\ x_2$	$f(s, x_1, x_2)$
m_0	0 0 0	0
m_1	0 0 1	0
m_2	0 1 0	1
m_3	0 1 1	1
m_4	1 0 0	0
m_5	1 0 1	1
m_6	1 1 0	0
m_7	1 1 1	1

x_2 \ $s\ x_1$	00	01	11	10
0	0	1	0	0
1	0	1	1	1

Let's Draw the K-map again

	$s\ x_1\ x_2$	$f(s, x_1, x_2)$
m_0	0 0 0	0
m_1	0 0 1	0
m_2	0 1 0	1
m_3	0 1 1	1
m_4	1 0 0	0
m_5	1 0 1	1
m_6	1 1 0	0
m_7	1 1 1	1



$$f(s, x_1, x_2) = \bar{s}\, x_1 + s\, x_2$$

This is correct!

Two Different Ways to Draw the K-map

x_1	x_2	x_3	
0	0	0	m_0
0	0	1	m_1
0	1	0	m_2
0	1	1	m_3
1	0	0	m_4
1	0	1	m_5
1	1	0	m_6
1	1	1	m_7

(a) Truth table

x_3 \ x_1x_2	00	01	11	10
0	m_0	m_2	m_6	m_4
1	m_1	m_3	m_7	m_5

(b) Karnaugh map

x_1 \ x_2x_3	00	01	11	10
0	m_0	m_1	m_3	m_2
1	m_4	m_5	m_7	m_6

Another Way to Draw 3-variable K-map

x_1	x_2	x_3	
0	0	0	m_0
0	0	1	m_1
0	1	0	m_2
0	1	1	m_3
1	0	0	m_4
1	0	1	m_5
1	1	0	m_6
1	1	1	m_7

(a) Truth table

x_3 \ x_1x_2	00	01	11	10
0	m_0	m_2	m_6	m_4
1	m_1	m_3	m_7	m_5

(b) Karnaugh map

x_2x_3 \ x_1	0	1
00	m_0	m_4
01	m_1	m_5
11	m_3	m_7
10	m_2	m_6

Gray Code

- **Sequence of binary codes**
- **Consecutive lines vary by only 1 bit**

	`000`
	`001`
`00`	`011`
`01`	`010`
`11`	`110`
`10`	`111`
	`101`
	`100`

Gray Code & K-map

	$s\,x_1\,x_2$
m_0	0 0 0
m_1	0 0 1
m_2	0 1 0
m_3	0 1 1
m_4	1 0 0
m_5	1 0 1
m_6	1 1 0
m_7	1 1 1

x_2 \ $s\,x_1$	00	01	11	10
0	m_0	m_2	m_6	m_4
1	m_1	m_3	m_7	m_5

Gray Code & K-map

	$s\ x_1\ x_2$
m_0	0 0 0
m_1	0 0 1
m_2	0 1 0
m_3	0 1 1
m_4	1 0 0
m_5	1 0 1
m_6	1 1 0
m_7	1 1 1

x_2 \ $s\ x_1$	00	01	11	10
0	000	010	110	100
1	001	011	111	101

Gray Code & K-map

	$s\ x_1\ x_2$
m_0	0 0 0
m_1	0 0 1
m_2	0 1 0
m_3	0 1 1
m_4	1 0 0
m_5	1 0 1
m_6	1 1 0
m_7	1 1 1

x_2 \ $s\ x_1$	00	01	11	10
0	000	010	110	100
1	001	011	111	101

These two neighbors
differ only in the LAST bit

Gray Code & K-map

	$s\ x_1\ x_2$
m_0	0 0 0
m_1	0 0 1
m_2	0 1 0
m_3	0 1 1
m_4	1 0 0
m_5	1 0 1
m_6	1 1 0
m_7	1 1 1

x_2 \ $s\ x_1$	00	01	11	10
0	000	010	110	100
1	001	011	111	101

These two neighbors differ only in the LAST bit

Gray Code & K-map

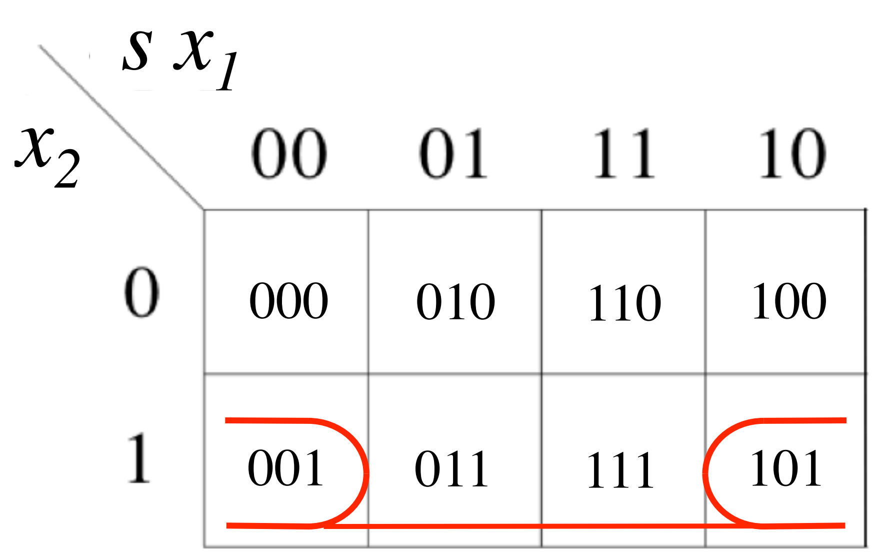
	$s\ x_1\ x_2$
m_0	0 0 0
m_1	0 0 1
m_2	0 1 0
m_3	0 1 1
m_4	1 0 0
m_5	1 0 1
m_6	1 1 0
m_7	1 1 1

x_2 \ $s\ x_1$	00	01	11	10
0	000	010	110	100
1	001	011	111	101

These two neighbors
differ only in the FIRST bit

Gray Code & K-map

	$s\ x_1\ x_2$
m_0	0 0 0
m_1	0 0 1
m_2	0 1 0
m_3	0 1 1
m_4	1 0 0
m_5	1 0 1
m_6	1 1 0
m_7	1 1 1



These two neighbors
differ only in the FIRST bit

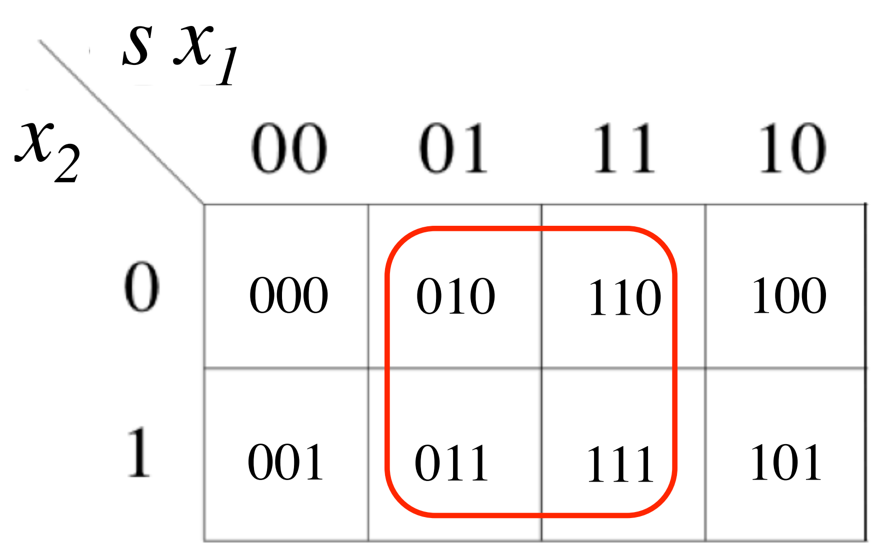
Adjacency Rules

x_2 \ $s\,x_1$	00	01	11	10
0	000	010	110	100
1	001	011	111	101

adjacent
columns

Gray Code & K-map

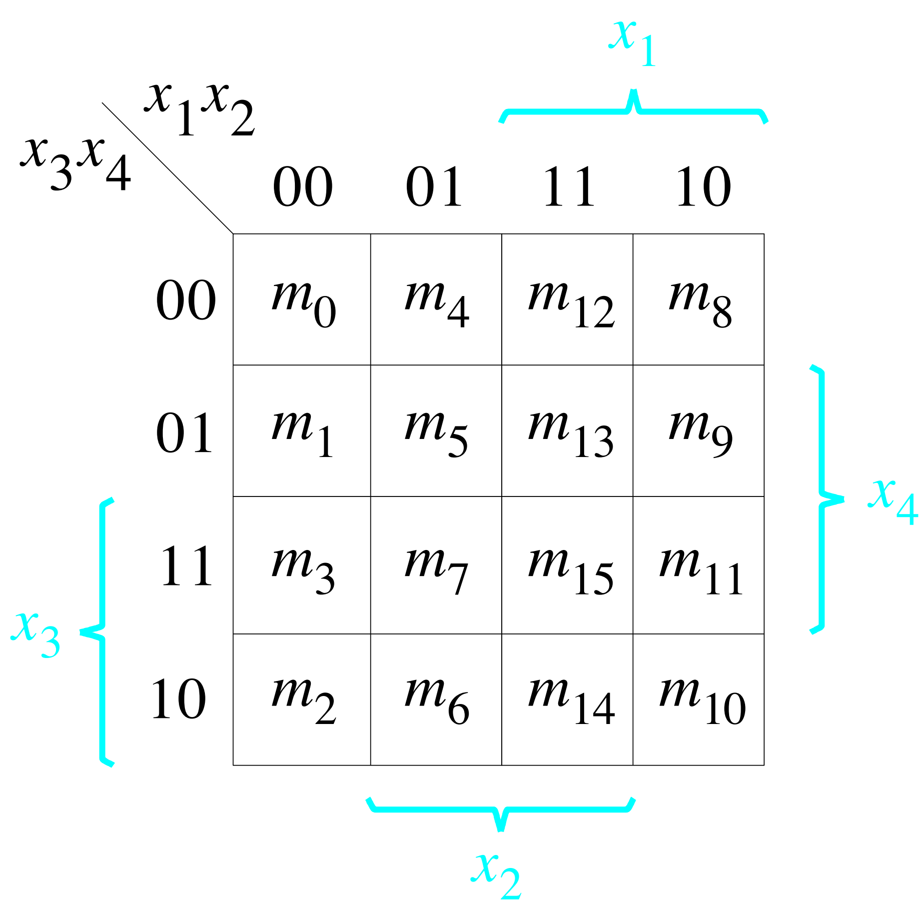
	$s\ x_1\ x_2$
m_0	0 0 0
m_1	0 0 1
m_2	0 1 0
m_3	0 1 1
m_4	1 0 0
m_5	1 0 1
m_6	1 1 0
m_7	1 1 1



These four neighbors
differ in the FIRST and LAST bit

They are similar in their MIDDLE bit

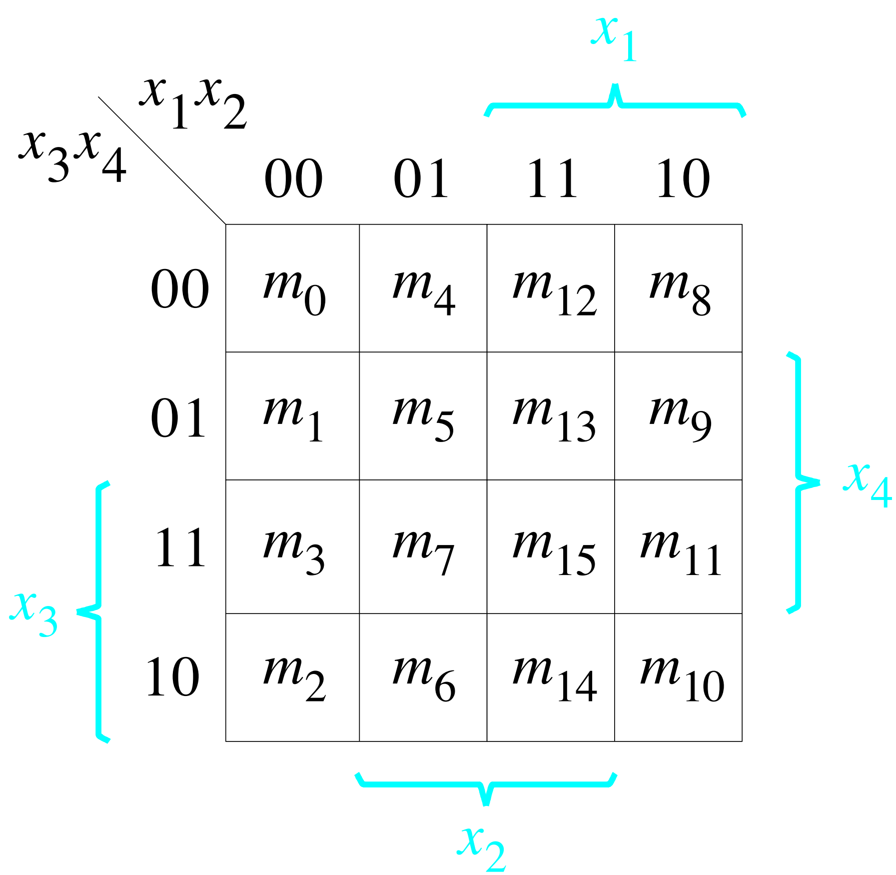
A four-variable Karnaugh map



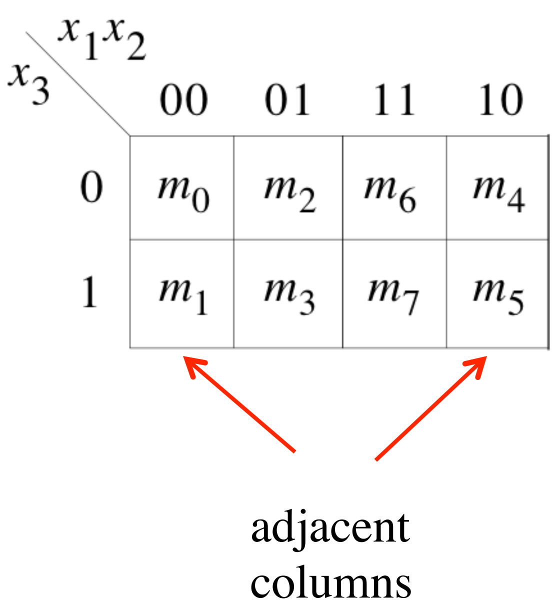
[Figure 2.53 from the textbook]

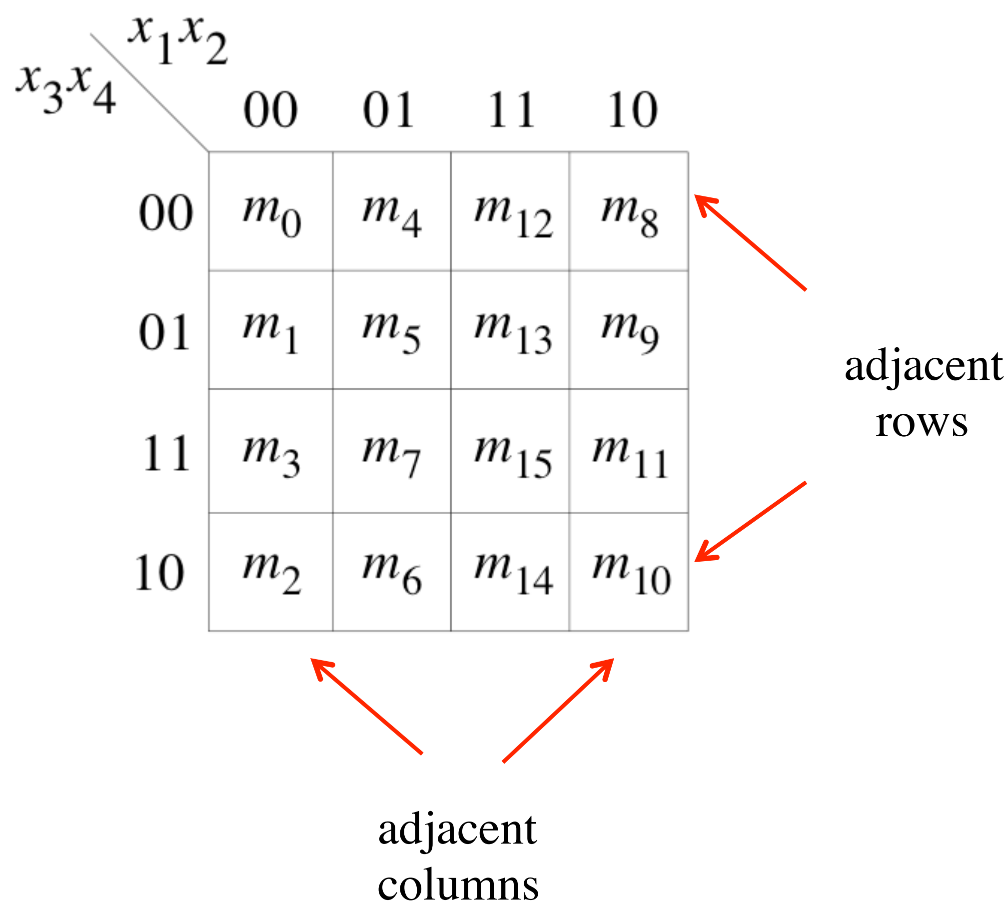
A four-variable Karnaugh map

x1	x2	x3	x4	
0	0	0	0	m0
0	0	0	1	m1
0	0	1	0	m2
0	0	1	1	m3
0	1	0	0	m4
0	1	0	1	m5
0	1	1	0	m6
0	1	1	1	m7
1	0	0	0	m8
1	0	0	1	m9
1	0	1	0	m10
1	0	1	1	m11
1	1	0	0	m12
1	1	0	1	m13
1	1	1	0	m14
1	1	1	1	m15



Adjacency Rules





Gray Code & K-map

x1	x2	x3	x4	
0	0	0	0	m0
0	0	0	1	m1
0	0	1	0	m2
0	0	1	1	m3
0	1	0	0	m4
0	1	0	1	m5
0	1	1	0	m6
0	1	1	1	m7
1	0	0	0	m8
1	0	0	1	m9
1	0	1	0	m10
1	0	1	1	m11
1	1	0	0	m12
1	1	0	1	m13
1	1	1	0	m14
1	1	1	1	m15

x_3x_4 \ x_1x_2	00	01	11	10
00	m_0	m_4	m_{12}	m_8
01	m_1	m_5	m_{13}	m_9
11	m_3	m_7	m_{15}	m_{11}
10	m_2	m_6	m_{14}	m_{10}

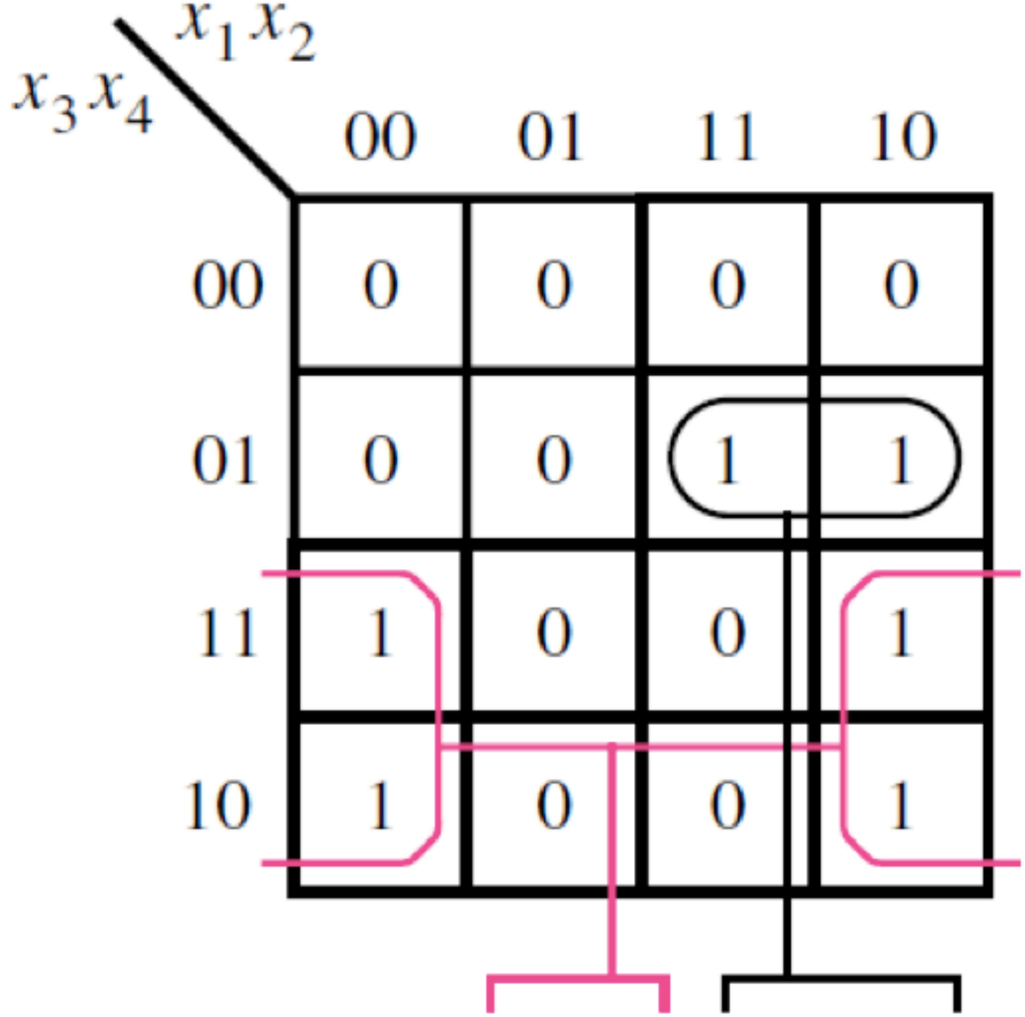
x_1 (columns 11, 10); x_2 (columns 01, 11); x_3 (rows 11, 10); x_4 (rows 01, 11)

Gray Code & K-map

x1	x2	x3	x4	
0	0	0	0	m0
0	0	0	1	m1
0	0	1	0	m2
0	0	1	1	m3
0	1	0	0	m4
0	1	0	1	m5
0	1	1	0	m6
0	1	1	1	m7
1	0	0	0	m8
1	0	0	1	m9
1	0	1	0	m10
1	0	1	1	m11
1	1	0	0	m12
1	1	0	1	m13
1	1	1	0	m14
1	1	1	1	m15

x_3x_4 \ x_1x_2	00	01	11	10
00	0000	0100	1100	1000
01	0001	0101	1101	1001
11	0011	0111	1111	1011
10	0010	0110	1110	1010

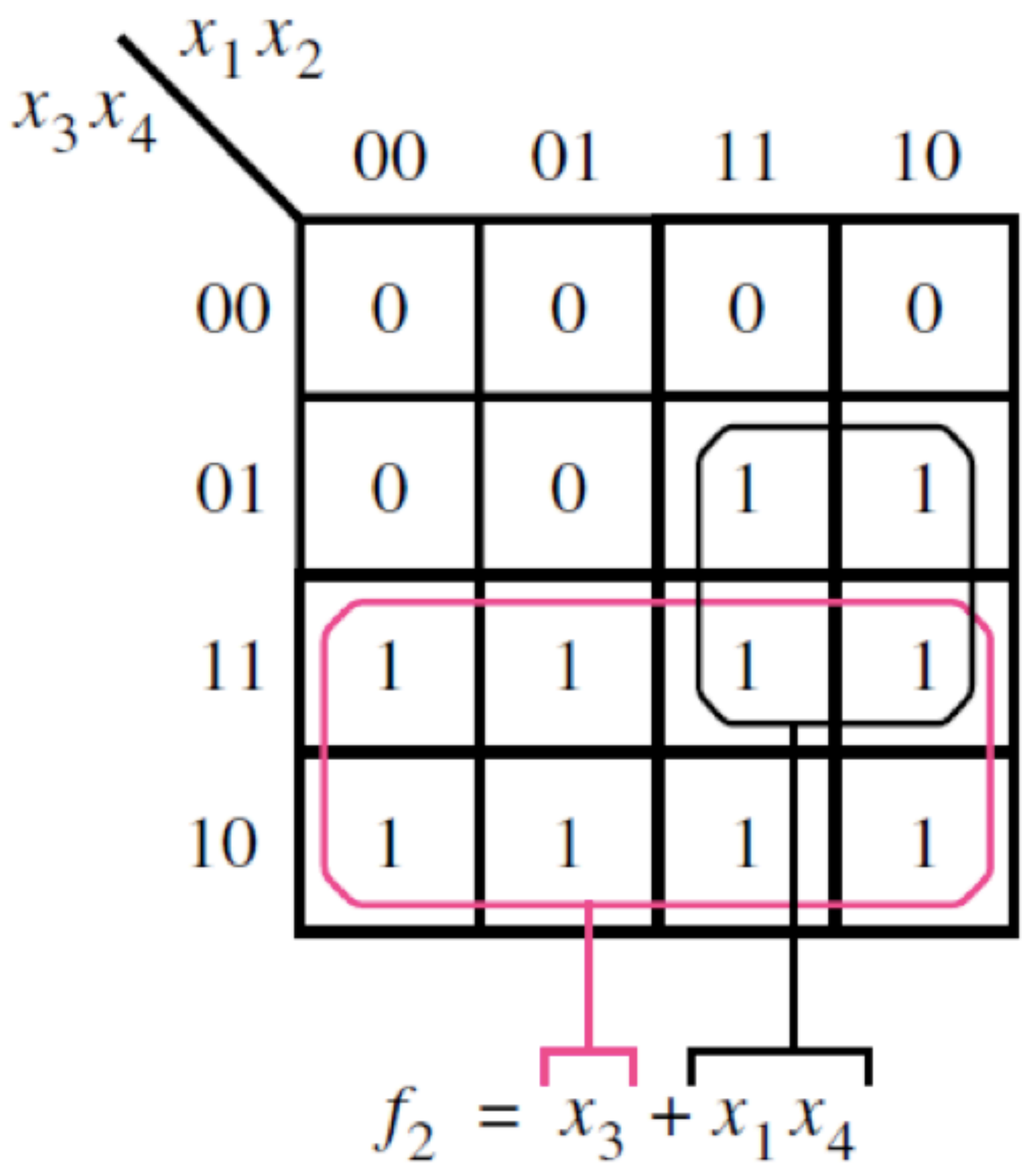
Example of a four-variable Karnaugh map



$$f_1 = \bar{x}_2 x_3 + x_1 \bar{x}_3 x_4$$

[Figure 2.54 from the textbook]

Example of a four-variable Karnaugh map



[Figure 2.54 from the textbook]

Strategy For Minimization

Grouping Rules

- Group "1"s with rectangles
- Both sides a power of 2:
  - 1x1, 1x2, 2x1, 2x2, 1x4, 4x1, 2x4, 4x2, 4x4
- Can use the same minterm more than once
- Can wrap around the edges of the map
- Some rules in selecting groups:
  - Try to use as few groups as possible to cover all "1"s.
  - For each group, try to make it as large as you can (i.e., if you can use a 2x2, don't use a 2x1 even if that is enough).

Terminology

Literal: a variable, complemented or uncomplemented

Some Examples:

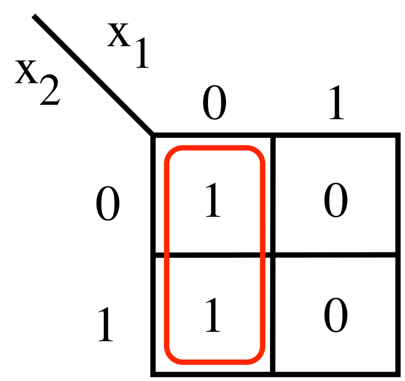
- $\overline{X_1}$
- X_2

Terminology

- **Implicant: product term that indicates the input combinations for which function output is 1**

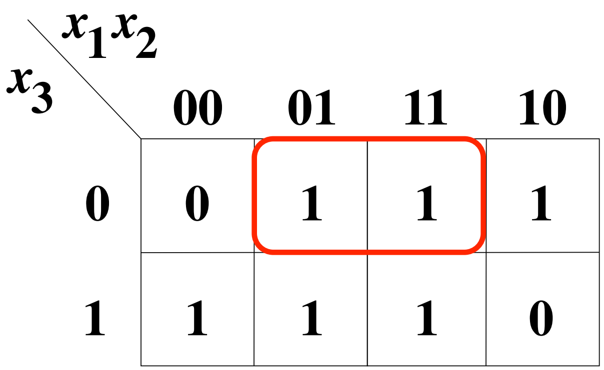
- **Example**

  - $\bar{x}_1$ **- indicates that $\bar{x}_1\bar{x}_2$ and $\bar{x}_1 x_2$ yield output of 1**

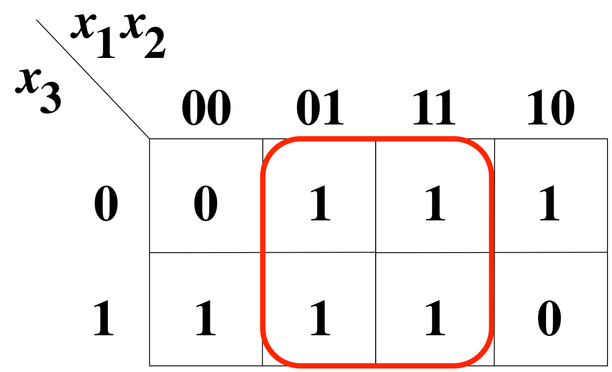


Terminology

- **Prime Implicant**
  - **Implicant that cannot be combined into another implicant with fewer literals**
  - **Some Examples**



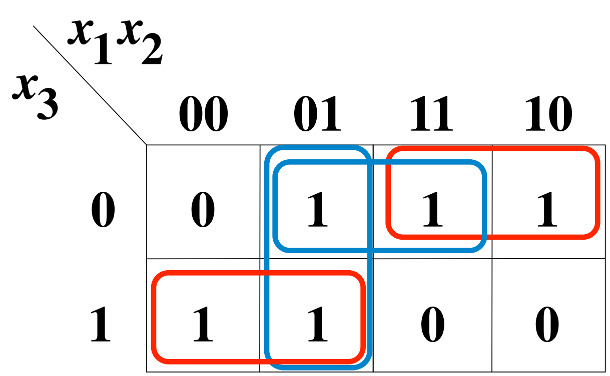
Not prime



Prime

Terminology

- **Essential Prime Implicant**
  - **Prime implicant that includes a minterm not covered by any other prime implicant**
  - **Some Examples**



Terminology

- **Cover**
  - **Collection of implicants that account for all possible input valuations where output is 1**

  - **Ex.** $x_1' x_2 x_3 + x_1 x_2 x_3' + x_1 x_2' x_3'$
  - **Ex.** $x_1' x_2 x_3 + x_1 x_3'$

x_3 \ x_1x_2	00	01	11	10
0	0	0	1	1
1	0	1	0	0

Example

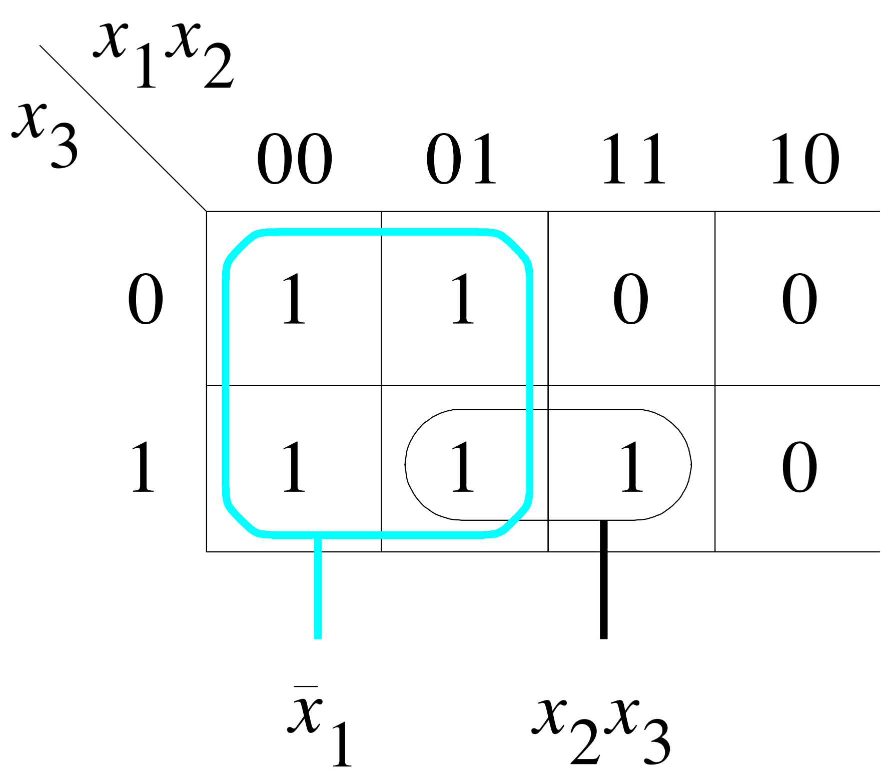
- **Give the Number of**
  - **Implicants?**
  - **Prime Implicants?**
  - **Essential Prime Implicants?**

x_3 \ $x_1 x_2$	00	01	11	10
0	1	1	0	0
1	1	1	1	0

Why concerned with minimization?

- Simplified function
- Reduce the cost of the circuit
  - Cost: Gates + Inputs
  - Transistors

Three-variable function $f(x_1, x_2, x_3) = \Sigma m(0, 1, 2, 3, 7)$

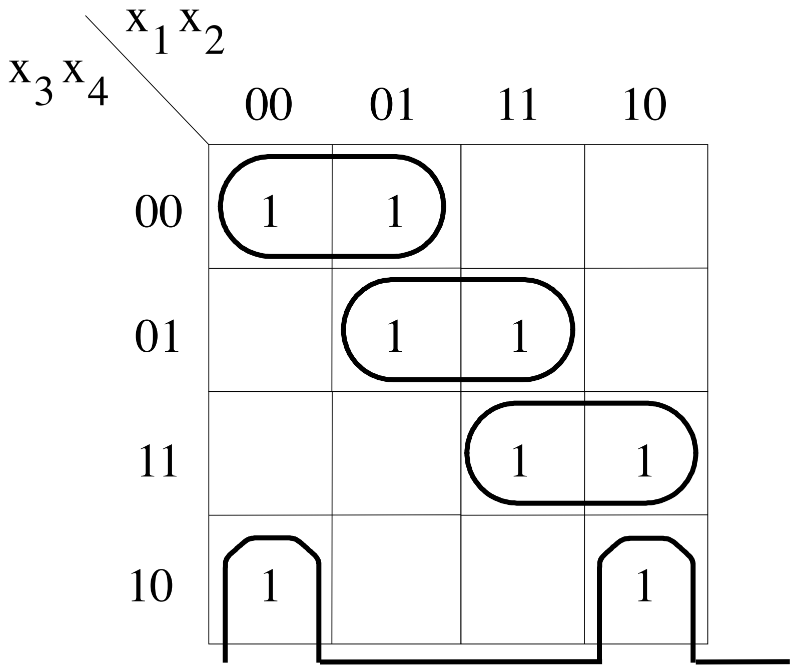


[Figure 2.56 from the textbook]

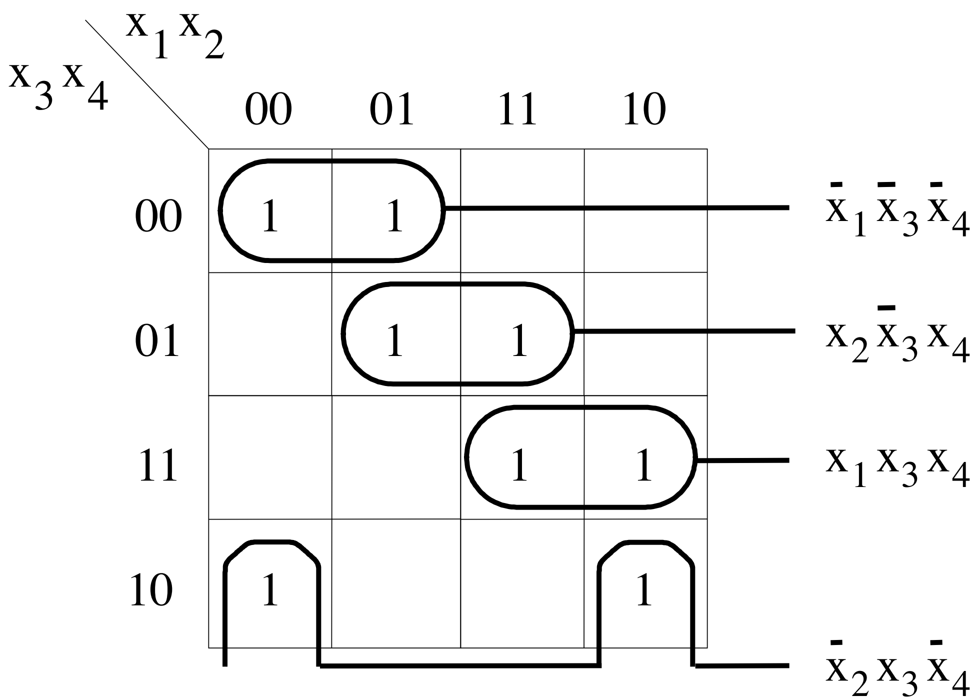
Example

x_3x_4 \ x_1x_2	00	01	11	10
00	1	1		
01		1	1	
11			1	1
10	1			1

Example

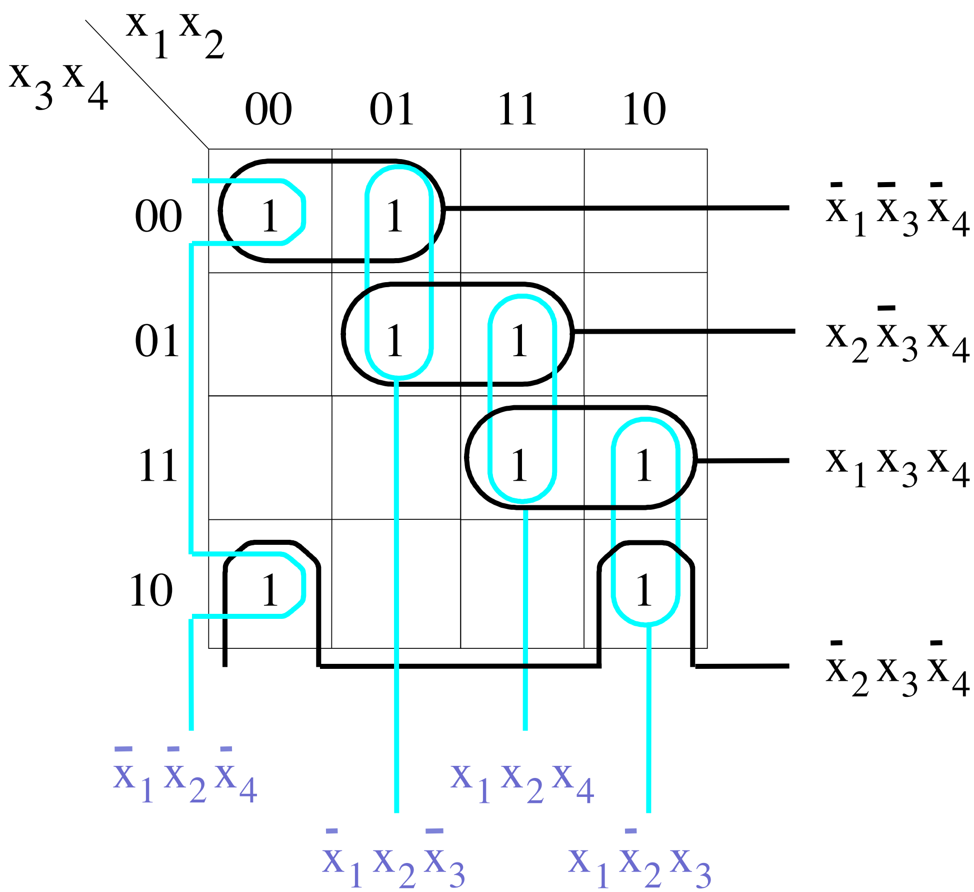


Example



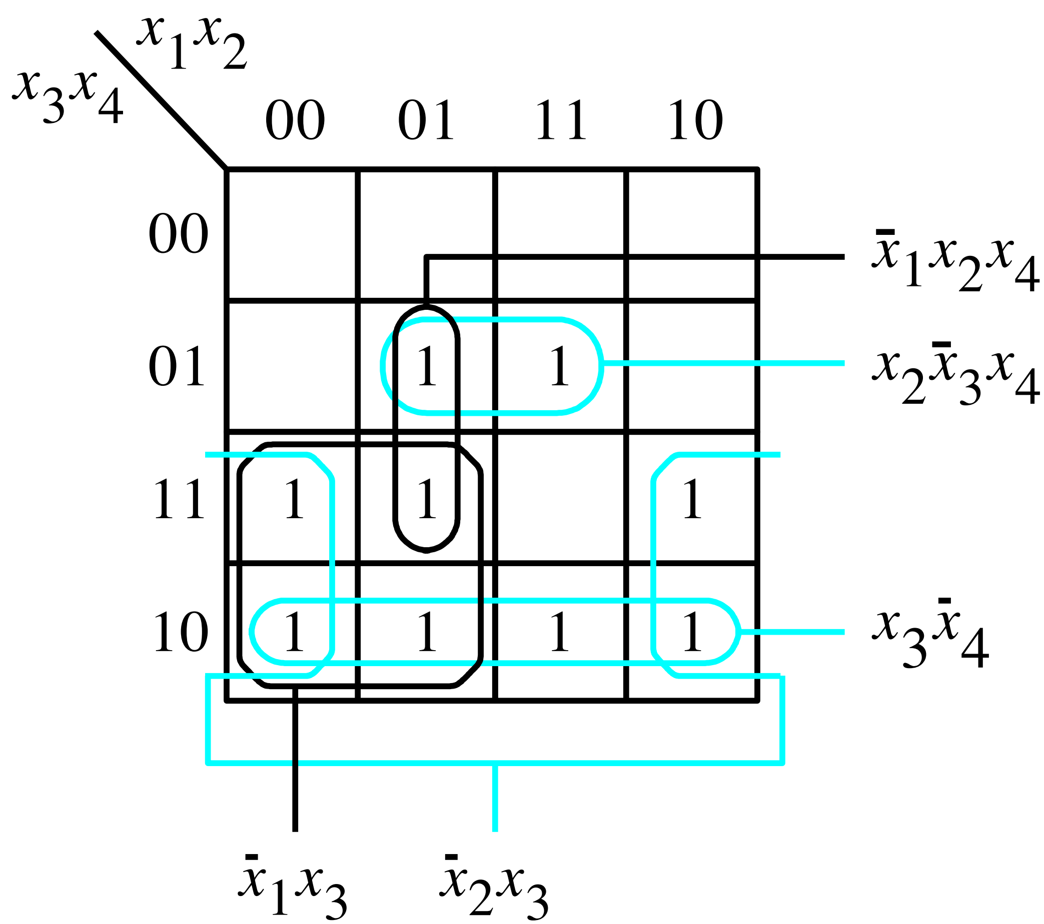
$x_1 x_2$
$x_3 x_4$
00
01
11
10
00
01
11
10
1
1
1
1
1
1
1
1
$\bar{x}_1 \bar{x}_3 \bar{x}_4$
$x_2 \bar{x}_3 x_4$
$x_1 x_3 x_4$
$\bar{x}_2 x_3 \bar{x}_4$

Example: Another Solution



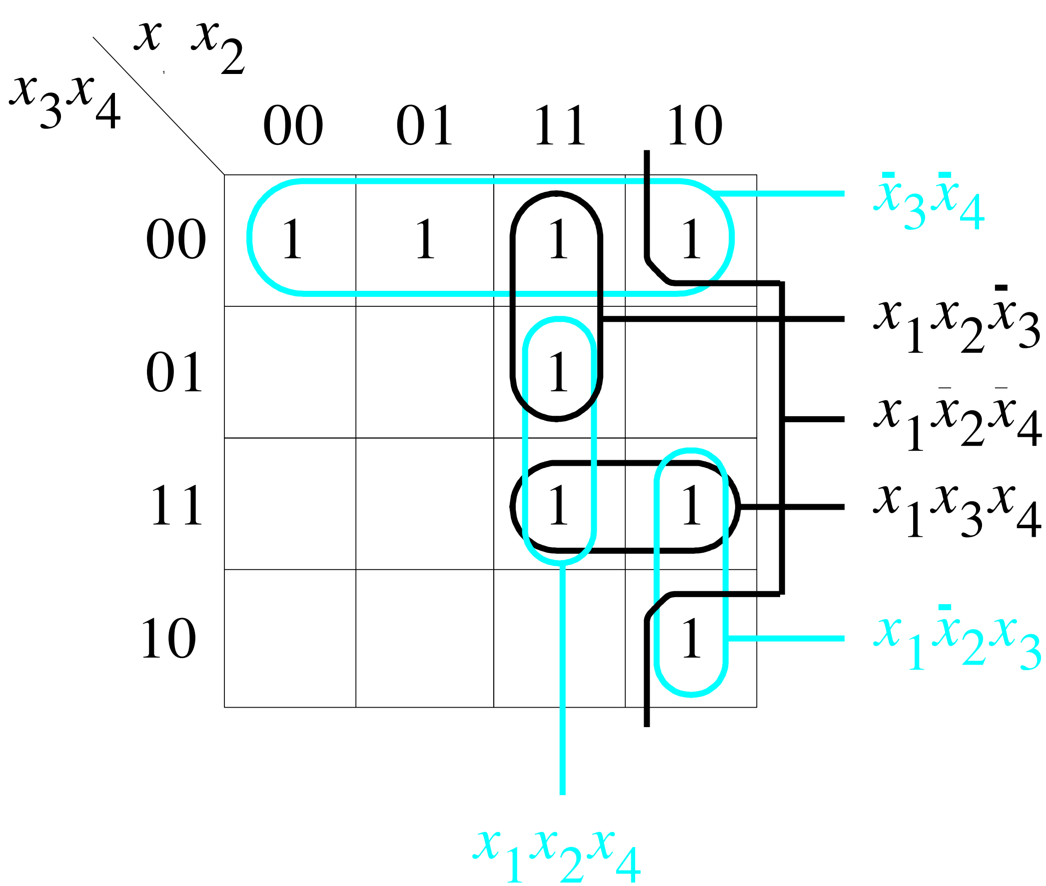
[Figure 2.59 from the textbook]

$$f(x_1,\ldots,x_4) = \Sigma\, m(2, 3, 5, 6, 7, 10, 11, 13, 14)$$



[Figure 2.57 from the textbook]

$$f(x_1, \ldots, x_4) = \Sigma\, m(0, 4, 8, 10, 11, 12, 13, 15)$$



[Figure 2.58 from the textbook]

Minimization of Product-of-Sums Forms

Do You Still Remember This Boolean Algebra Theorem?

14a. $x \bullet y + x \bullet \overline{y} = x$ Combining

14b. $(x + y) \bullet (x + \overline{y}) = x$

Let's prove 14.b

x	y	$(x + y) \cdot (x + \overline{y}) = x$
0	0	
0	1	
1	0	
1	1	

Let's prove 14.b

x	y	$(x + y) \cdot (x + \overline{y}) = x$
0	0	0
0	1	1
1	0	1
1	1	1

Let’s prove 14.b

x	y	$(x + y)\cdot(x + \overline{y}) = x$	
0	0	0	1
0	1	1	0
1	0	1	1
1	1	1	1

Let’s prove 14.b

x	y	$(x + y) \cdot (x + \bar{y}) = x$		
0	0	0	0	1
0	1	1	0	0
1	0	1	1	1
1	1	1	1	1

Let’s prove 14.b

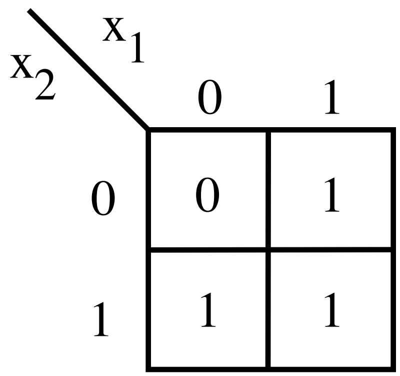
x	y	$(x + y)$	$\bullet$	$(x + \overline{y})$	$= x$
0	0	0	0	1	0
0	1	1	0	0	0
1	0	1	1	1	1
1	1	1	1	1	1

Let’s prove 14.b

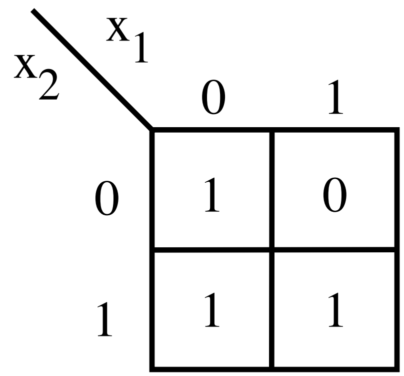
x	y	$(x + y)$	$\bullet$	$(x + \overline{y})$	$= x$
0	0	0	0	1	0
0	1	1	0	0	0
1	0	1	1	1	1
1	1	1	1	1	1

They are equal.

Grouping Example



M_0



M_2

Grouping Example

x_2 \ x_1	0	1
0	0	1
1	1	1

*

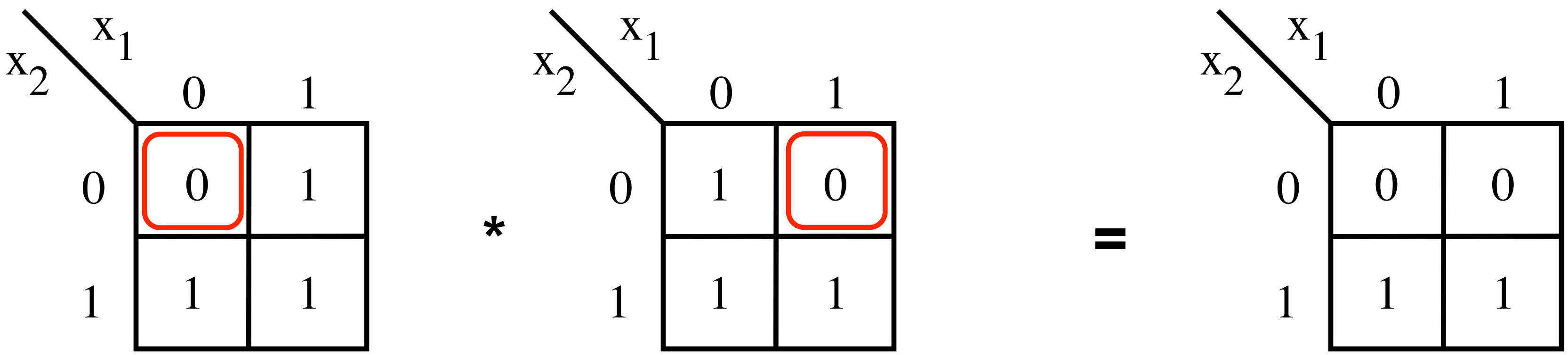
x_2 \ x_1	0	1
0	1	0
1	1	1

=

x_2 \ x_1	0	1
0	0	0
1	1	1

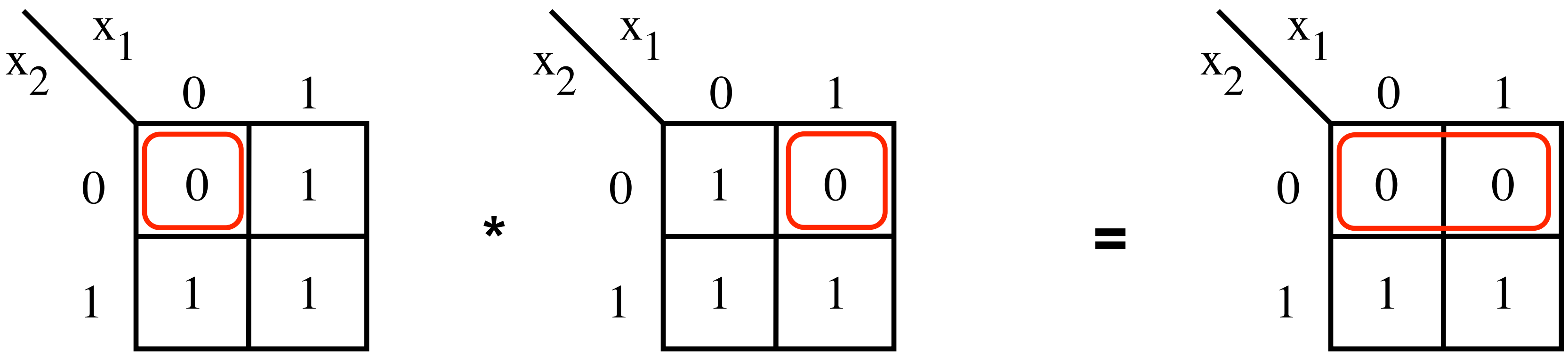
M_0 * M_2 = $M_0 * M_2$

Grouping Example



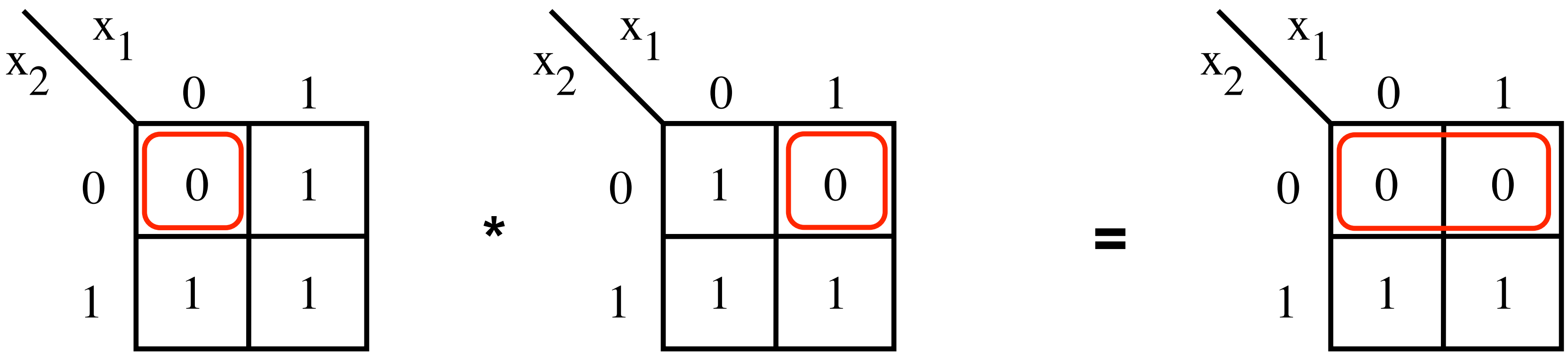
M_0 * M_2 = M_0 * M_2

Grouping Example



M_0 * M_2 = $M_0 * M_2$

Grouping Example

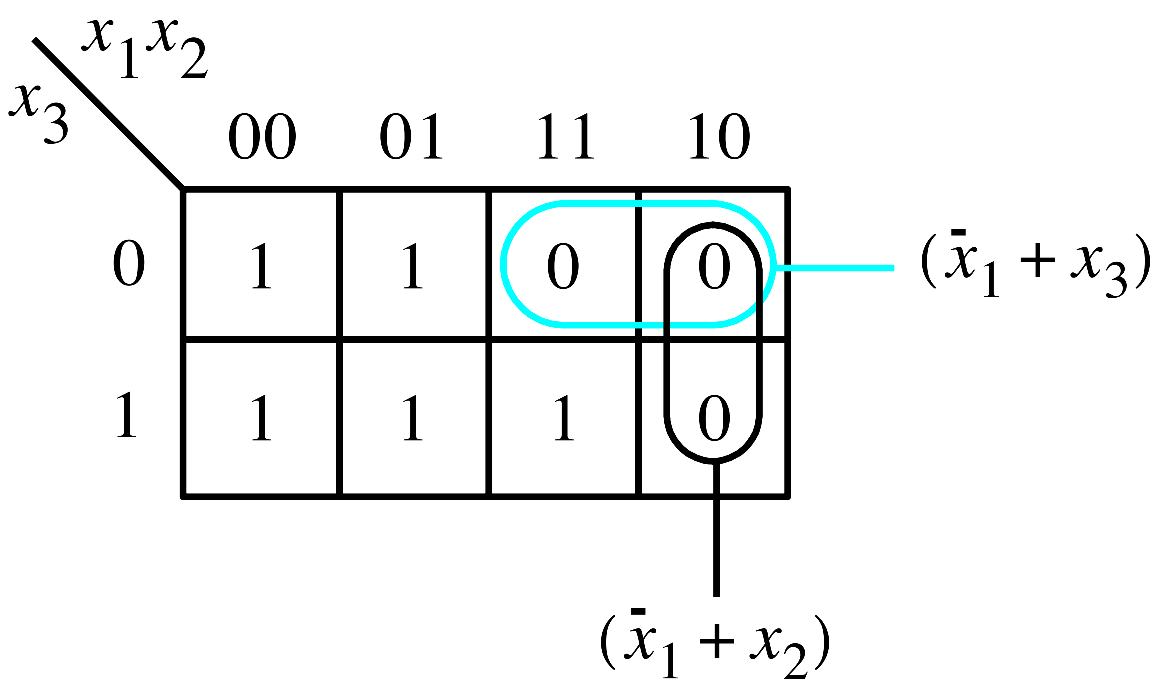


$$M_0 \quad * \quad M_2 \quad = \quad M_0 * M_2$$

$$(x_1 + x_2) \quad * \quad (\overline{x}_1 + x_2) \quad = \quad x_2$$

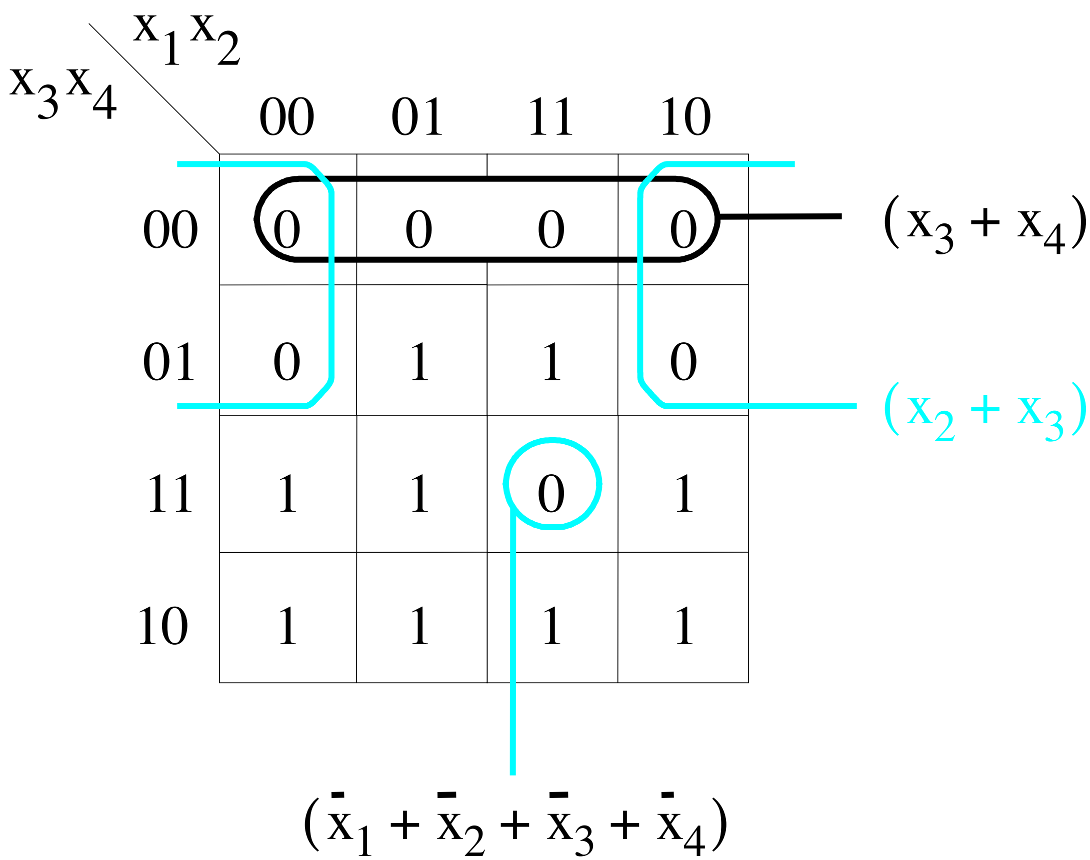
Property 14b (Combining)

POS minimization of $f(x_1, x_2, x_3) = \Pi\ M(4, 5, 6)$



[Figure 2.60 from the textbook]

POS minimization of $f(x_1,\ldots,x_4) = \Pi M(0, 1, 4, 8, 9, 12, 15)$



[Figure 2.61 from the textbook]

Questions?

THE END