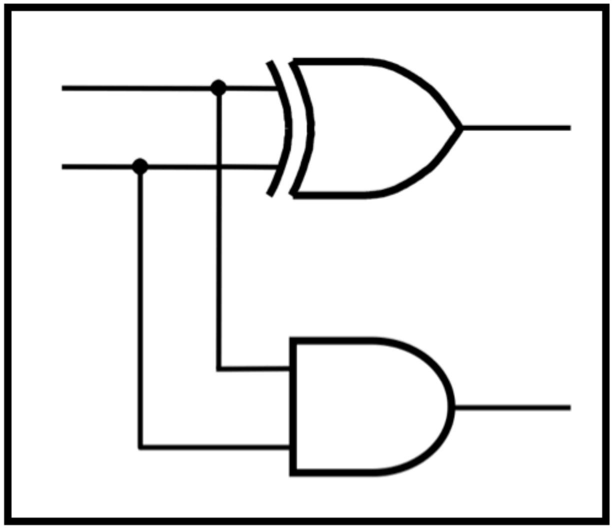


CprE 281: Digital Logic

Instructor: Alexander Stoytchev

http://www.ece.iastate.edu/~alexs/classes/

Boolean Algebra

CprE 281: Digital Logic
Iowa State University, Ames, IA

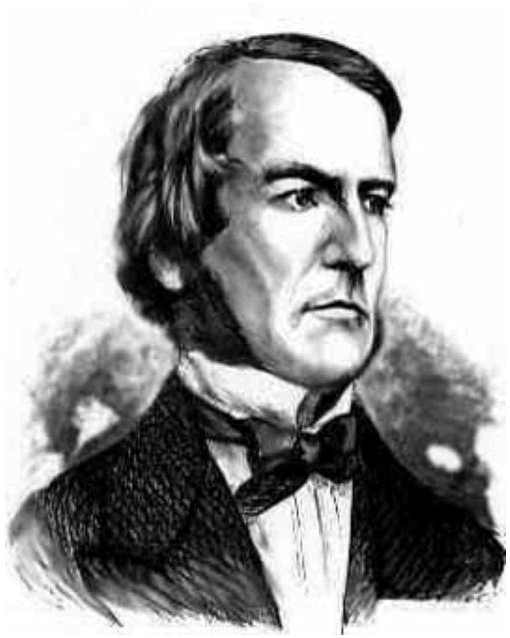
Administrative Stuff

- **HW1 is due today**

Administrative Stuff

- HW2 is out
- It is due on Wednesday Sep 5 @ 4pm.
- Submit it on paper before the start of the lecture

Boolean Algebra



George Boole
1815-1864

- **An algebraic structure consists of**
  - **a set of elements {0, 1}**
  - **binary operators {+, •}**
  - **and a unary operator { ' } or { ¯ }**

- **Introduced by George Boole in 1854**

- **An effective means of describing circuits built with switches**
- **A powerful tool that can be used for designing and analyzing logic circuits**

Axioms of Boolean Algebra

1a. $0 \bullet 0 = 0$

1b. $1 + 1 = 1$

2a. $1 \bullet 1 = 1$

2b. $0 + 0 = 0$

3a. $0 \bullet 1 = 1 \bullet 0 = 0$

3b. $1 + 0 = 0 + 1 = 1$

4a. If $x=0$, then $\overline{x} = 1$

4b. If $x=1$, then $\overline{x} = 0$

Single-Variable Theorems

5a. $x \bullet 0 = 0$

5b. $x + 1 = 1$

6a. $x \bullet 1 = x$

6b. $x + 0 = x$

7a. $x \bullet x = x$

7b. $x + x = x$

8a. $x \bullet \overline{x} = 0$

8b. $x + \overline{x} = 1$

9. $\overline{\overline{x}} = x$

Two- and Three-Variable Properties

10a.	$x \bullet y = y \bullet x$	Commutative
10b.	$x + y = y + x$	
11a.	$x \bullet (y \bullet z) = (x \bullet y) \bullet z$	Associative
11b.	$x + (y + z) = (x + y) + z$	
12a.	$x \bullet (y + z) = x \bullet y + x \bullet z$	Distributive
12b.	$x + y \bullet z = (x + y) \bullet (x + z)$	
13a.	$x + x \bullet y = x$	Absorption
13b.	$x \bullet (x + y) = x$	

Two- and Three-Variable Properties

14a.	$x \bullet y + x \bullet \overline{y} = x$	Combining
14b.	$(x + y)\bullet(x + \overline{y}) = x$	
15a.	$\overline{x \bullet y} = \overline{x} + \overline{y}$	DeMorgan's theorem
15b.	$\overline{x + y} = \overline{x} \bullet \overline{y}$	
16a.	$x + \overline{x}\bullet y = x + y$	
16b.	$x\bullet(\overline{x} + y) = x\bullet y$	
17a.	$x\bullet y + y\bullet z + \overline{x}\bullet z = x\bullet y + \overline{x}\bullet z$	Consensus
17b.	$(x+y)\bullet(y+z)\bullet(\overline{x}+z)=(x+y)\bullet(\overline{x}+z)$	

Now, let’s prove all of these

The First Four are Axioms (i.e., they don't require a proof)

1a. $0 \bullet 0 = 0$

1b. $1 + 1 = 1$

2a. $1 \bullet 1 = 1$

2b. $0 + 0 = 0$

3a. $0 \bullet 1 = 1 \bullet 0 = 0$

3b. $1 + 0 = 0 + 1 = 1$

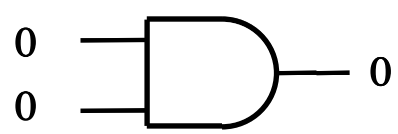
4a. If $x=0$, then $\overline{x} = 1$

4b. If $x=1$, then $\overline{x} = 0$

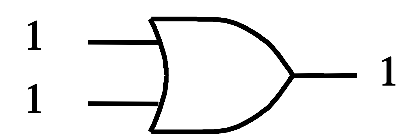
But here are some other ways to think about them

1a. $0 \cdot 0 = 0$

1b. $1 + 1 = 1$



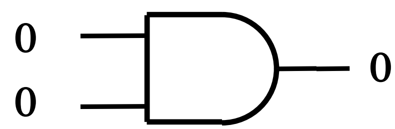
AND gate



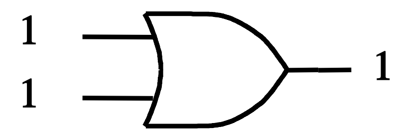
OR gate

1a. $0 \cdot 0 = 0$

1b. $1 + 1 = 1$



AND gate



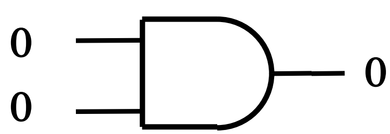
OR gate

x_1	x_2	f
0	0	0
0	1	0
1	0	0
1	1	1

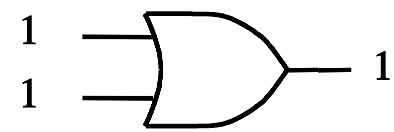
x_1	x_2	f
0	0	0
0	1	1
1	0	1
1	1	1

1a. $0 \cdot 0 = 0$

1b. $1 + 1 = 1$



AND gate



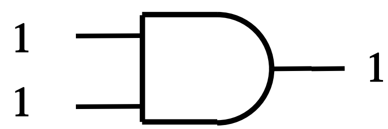
OR gate

x_1	x_2	f
0	0	0
0	1	0
1	0	0
1	1	1

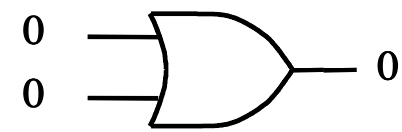
x_1	x_2	f
0	0	0
0	1	1
1	0	1
1	1	1

2a. $1 \bullet 1 = 1$

2b. $0 + 0 = 0$

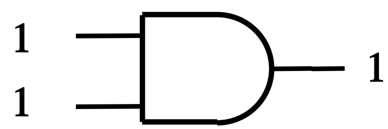


AND gate



OR gate

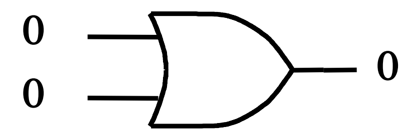
2a. $1 \cdot 1 = 1$



AND gate

x_1	x_2	f
0	0	0
0	1	0
1	0	0
1	1	1

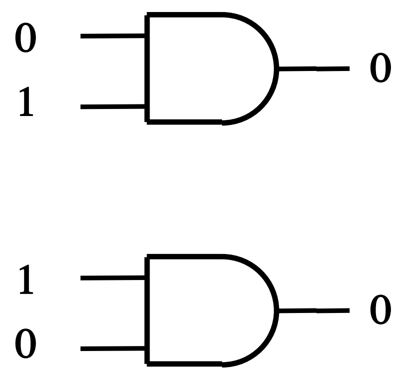
2b. $0 + 0 = 0$



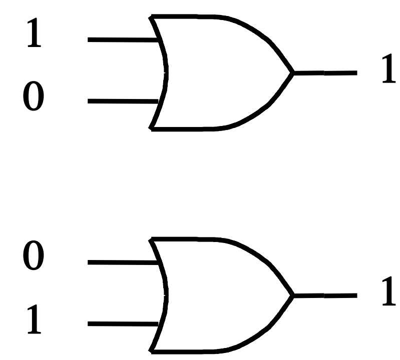
OR gate

x_1	x_2	f
0	0	0
0	1	1
1	0	1
1	1	1

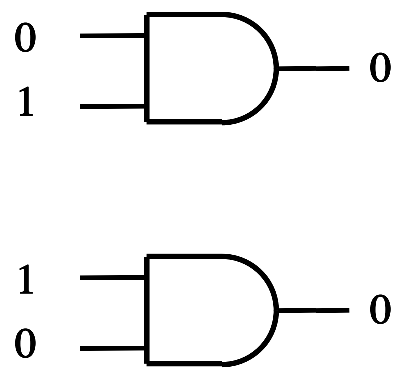
3a. 0 • 1 = 1 • 0 = 0



3b. 1 + 0 = 0 + 1 = 1



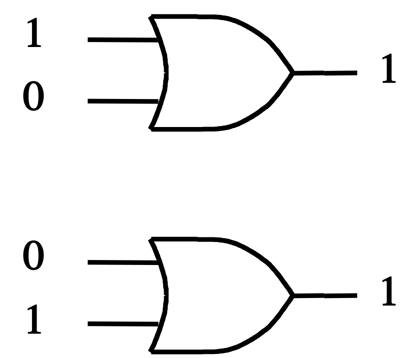
3a. $0 \cdot 1 = 1 \cdot 0 = 0$



AND gate

x_1	x_2	f
0	0	0
0	1	0
1	0	0
1	1	1

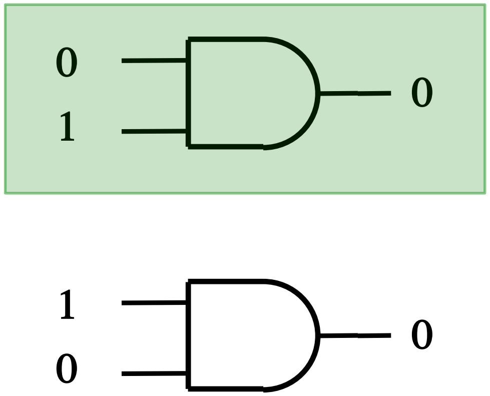
3b. $1 + 0 = 0 + 1 = 1$



OR gate

x_1	x_2	f
0	0	0
0	1	1
1	0	1
1	1	1

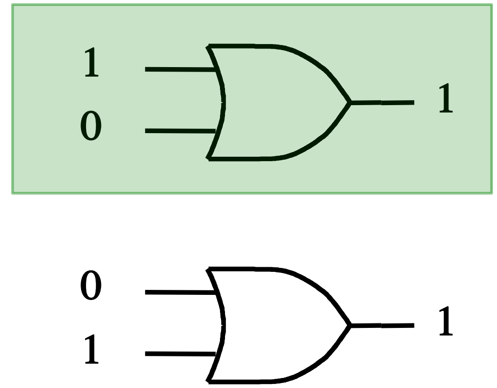
3a. $0 \cdot 1 = 1 \cdot 0 = 0$



AND gate

x_1	x_2	f
0	0	0
0	1	0
1	0	0
1	1	1

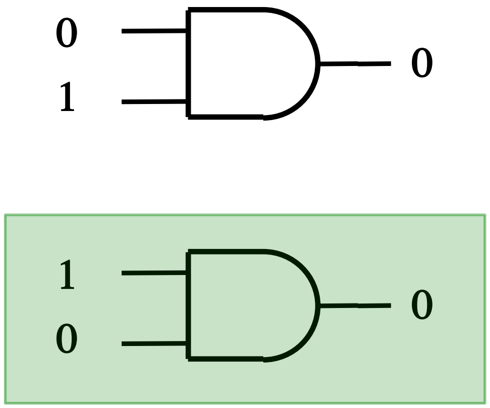
3b. $1 + 0 = 0 + 1 = 1$



OR gate

x_1	x_2	f
0	0	0
0	1	1
1	0	1
1	1	1

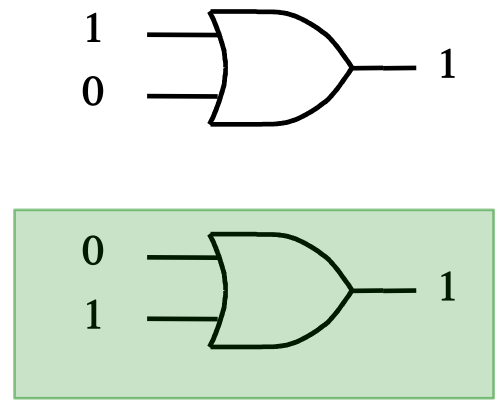
3a. $0 \cdot 1 = 1 \cdot 0 = 0$



AND gate

x_1	x_2	f
0	0	0
0	1	0
1	0	0
1	1	1

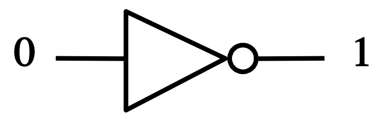
3b. $1 + 0 = 0 + 1 = 1$



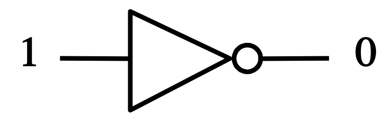
OR gate

x_1	x_2	f
0	0	0
0	1	1
1	0	1
1	1	1

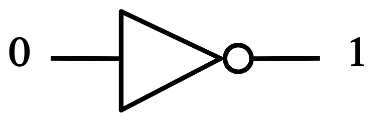
4a. If x=0, then $\overline{x} = 1$



4b. If x=1, then $\overline{x} = 0$



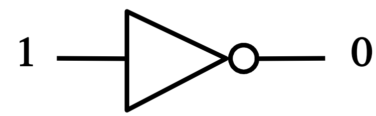
4a. If x=0, then $\overline{x}$ = 1



NOT gate

x	$\overline{x}$
0	1
1	0

4b. If x=1, then $\overline{x}$ = 0



NOT gate

x	$\overline{x}$
0	1
1	0

Single-Variable Theorems

5a. $x \bullet 0 = 0$

5b. $x + 1 = 1$

6a. $x \bullet 1 = x$

6b. $x + 0 = x$

7a. $x \bullet x = x$

7b. $x + x = x$

8a. $x \bullet \overline{x} = 0$

8b. $x + \overline{x} = 1$

9. $\overline{\overline{x}} = x$

5a. $x \cdot 0 = 0$

5a. $x \cdot 0 = 0$

The Boolean variable x can have only two possible values: 0 or 1. Let’s look at each case separately.

5a. $x \bullet 0 = 0$

The Boolean variable x can have only two possible values: 0 or 1. Let's look at each case separately.

i) If x = 0, then we have

$$0 \bullet 0 = 0 \qquad \text{// axiom 1a}$$

5a. $x \bullet 0 = 0$

The Boolean variable x can have only two possible values: 0 or 1. Let's look at each case separately.

i) If x = 0, then we have

$0 \bullet 0 = 0$ // axiom 1a

ii) If x = 1, then we have

$1 \bullet 0 = 0$ // axiom 3a

5b. $x + 1 = 1$

5b. $x + 1 = 1$

The Boolean variable x can have only two possible values: 0 or 1. Let's look at each case separately.

i) If $x = 0$, then we have

$0 + 1 = 1$ // axiom 3b

5b. $x + 1 = 1$

The Boolean variable x can have only two possible values: 0 or 1. Let's look at each case separately.

i) If $x = 0$, then we have

$0 + 1 = 1$ // axiom 3b

ii) If $x = 1$, then we have

$1 + 1 = 1$ // axiom 1b

6a. $x \cdot 1 = x$

The Boolean variable x can have only two possible values: 0 or 1. Let's look at each case separately.

i) If x = 0, then we have

$$0 \cdot 1 = 0$$ // axiom 3a

ii) If x = 1, then we have

$$1 \cdot 1 = 1$$ // axiom 2a

$$6a. \quad x \cdot 1 = x$$

The Boolean variable x can have only two possible values: 0 or 1. Let's look at each case separately.

i) If x = 0, then we have

$$0 \cdot 1 = 0 \qquad \text{// axiom 3a}$$

ii) If x = 1, then we have

$$1 \cdot 1 = 1 \qquad \text{// axiom 2a}$$

6b. $x + 0 = x$

The Boolean variable x can have only two possible values: 0 or 1. Let's look at each case separately.

i) If x = 0, then we have

$$0 + 0 = 0 \qquad \text{// axiom 2b}$$

ii) If x = 1, then we have

$$1 + 0 = 1 \qquad \text{// axiom 3b}$$

6b. $x + 0 = x$

The Boolean variable x can have only two possible values: 0 or 1. Let's look at each case separately.

i) If x = 0, then we have

$0 + 0 = 0$ // axiom 2b

ii) If x = 1, then we have

$1 + 0 = 1$ // axiom 3b

7a. $x \cdot x = x$

i) If $x = 0$, then we have

$$0 \cdot 0 = 0$$ // axiom 1a

ii) If $x = 1$, then we have

$$1 \cdot 1 = 1$$ // axiom 2a

7a. $x \bullet x = x$

i) If x = 0, then we have

$0 \bullet 0 = 0$ // axiom 1a

ii) If x = 1, then we have

$1 \bullet 1 = 1$ // axiom 2a

7b. $x + x = x$

i) If x = 0, then we have

$$0 + 0 = 0$$ // axiom 2b

ii) If x = 1, then we have

$$1 + 1 = 1$$ // axiom 1b

7b. $x + x = x$

i) If x = 0, then we have

$0 + 0 = 0$ // axiom 2b

ii) If x = 1, then we have

$1 + 1 = 1$ // axiom 1b

$$8a. \quad x \bullet \overline{x} = 0$$

i) If x = 0, then we have

$$0 \bullet 1 = 0 \qquad \text{// axiom 3a}$$

ii) If x = 1, then we have

$$1 \bullet 0 = 0 \qquad \text{// axiom 3a}$$

$$8a. \quad x \cdot \overline{x} = \boxed{0}$$

i) If x = 0, then we have

$$0 \cdot 1 = \boxed{0}$$ // axiom 3a

ii) If x = 1, then we have

$$1 \cdot 0 = \boxed{0}$$ // axiom 3a

8b. $x + \overline{x} = 1$

i) If x = 0, then we have

$$0 + 1 = 1$$ // axiom 3b

ii) If x = 1, then we have

$$1 + 0 = 1$$ // axiom 3b

8b. $x + \overline{x} = 1$

i) If x = 0, then we have

$0 + 1 = 1$ // axiom 3b

ii) If x = 1, then we have

$1 + 0 = 1$ // axiom 3b

9. $\overline{\overline{x}} = x$

i) If x = 0, then we have

$\overline{x} = 1$ // axiom 4a

let $y = \overline{x} = 1$, then we have

$\overline{y} = 0$ // axiom 4b

Therefore,

$\overline{\overline{x}} = x$ (when x =0)

9. $\overline{\overline{x}} = x$

ii) If x = 1, then we have

$\overline{x} = 0$ // axiom 4b

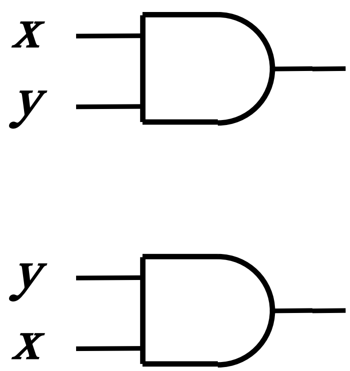
let $y = \overline{x} = 0$, then we have

$\overline{y} = 1$ // axiom 4a

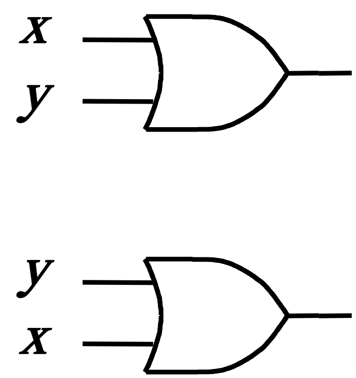
Therefore,

$\overline{\overline{x}} = x$ (when x =1)

10a. $x \cdot y = y \cdot x$

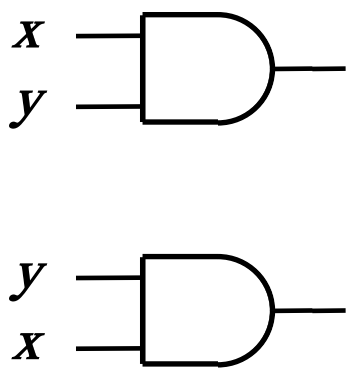


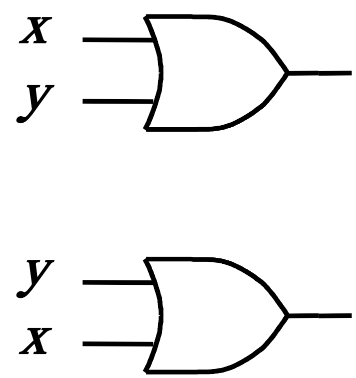
10b. $x + y = y + x$



10a. $x \cdot y = y \cdot x$

10b. $x + y = y + x$





AND gate

x	y	f
0	0	0
0	1	0
1	0	0
1	1	1

OR gate

x	y	f
0	0	0
0	1	1
1	0	1
1	1	1

The order of the inputs does not matter.

11a. $x \bullet (y \bullet z) = (x \bullet y) \bullet z$

x	y	z	x	y • z	x•(y•z)
0	0	0			
0	0	1			
0	1	0			
0	1	1			
1	0	0			
1	0	1			
1	1	0			
1	1	1			

Truth table for the left-hand side

11a. $x \bullet (y \bullet z) = (x \bullet y) \bullet z$

x	y	z	x	y • z	x•(y•z)
0	0	0	0	0	
0	0	1	0	0	
0	1	0	0	0	
0	1	1	0	1	
1	0	0	1	0	
1	0	1	1	0	
1	1	0	1	0	
1	1	1	1	1	

Truth table for the left-hand side

11a. $x \cdot (y \cdot z) = (x \cdot y) \cdot z$

x	y	z	x	y • z	x•(y•z)
0	0	0	0	0	0
0	0	1	0	0	0
0	1	0	0	0	0
0	1	1	0	1	0
1	0	0	1	0	0
1	0	1	1	0	0
1	1	0	1	0	0
1	1	1	1	1	1

Truth table for the left-hand side

11a. $x \bullet (y \bullet z) = (x \bullet y) \bullet z$

x	y	z	x • y	z	(x•y)•z
0	0	0	0	0	0
0	0	1	0	1	0
0	1	0	0	0	0
0	1	1	0	1	0
1	0	0	0	0	0
1	0	1	0	1	0
1	1	0	1	0	0
1	1	1	1	1	1

Truth table for the right-hand side

11a. $x \cdot (y \cdot z) = (x \cdot y) \cdot z$

x•(y•z)
0
0
0
0
0
0
0
1

(x•y)•z
0
0
0
0
0
0
0
1

These two are identical, which concludes the proof.

11b. $x + (y + z) = (x + y) + z$

x	y	z	x	y + z	x+(y+z)
0	0	0			
0	0	1			
0	1	0			
0	1	1			
1	0	0			
1	0	1			
1	1	0			
1	1	1			

Truth table for the left-hand side

11b. x + (y + z) = (x + y) + z

x	y	z	x	y + z	x+(y+z)
0	0	0	0	0	
0	0	1	0	1	
0	1	0	0	1	
0	1	1	0	1	
1	0	0	1	0	
1	0	1	1	1	
1	1	0	1	1	
1	1	1	1	1	

Truth table for the left-hand side

11b. $x + (y + z) = (x + y) + z$

x	y	z	x	y + z	x+(y+z)
0	0	0	0	0	0
0	0	1	0	1	1
0	1	0	0	1	1
0	1	1	0	1	1
1	0	0	1	0	1
1	0	1	1	1	1
1	1	0	1	1	1
1	1	1	1	1	1

Truth table for the left-hand side

11b. $x + (y + z) = (x + y) + z$

x	y	z	x + y	z	(x+y)+z
0	0	0	0	0	0
0	0	1	0	1	1
0	1	0	1	0	1
0	1	1	1	1	1
1	0	0	1	0	1
1	0	1	1	1	1
1	1	0	1	0	1
1	1	1	1	1	1

Truth table for the right-hand side

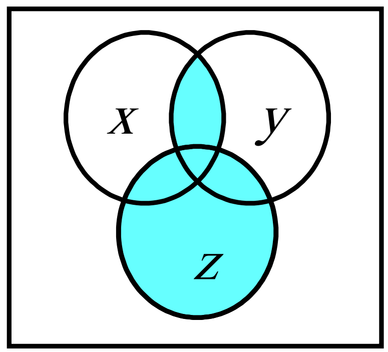
11b. $x + (y + z) = (x + y) + z$

x+(y+z)
0
1
1
1
1
1
1
1

(x+y)+z
0
1
1
1
1
1
1
1

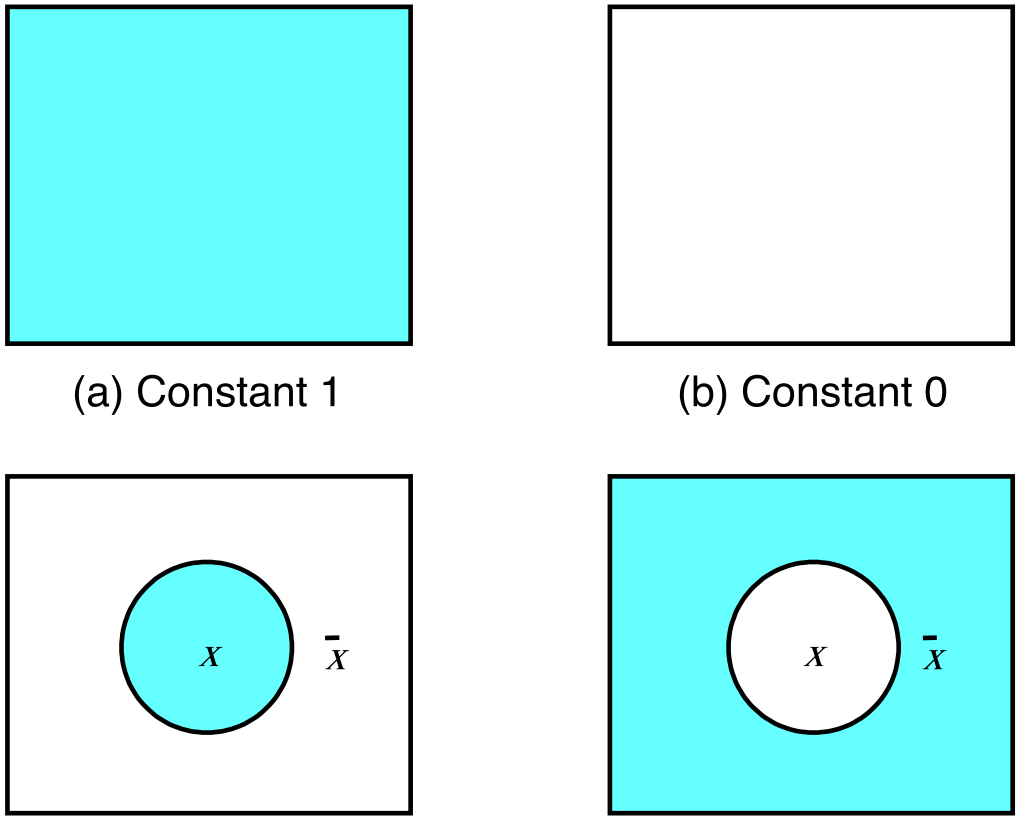
These two are identical, which concludes the proof.

The Venn Diagram Representation



$xy + z$

Venn Diagram Basics



(a) Constant 1

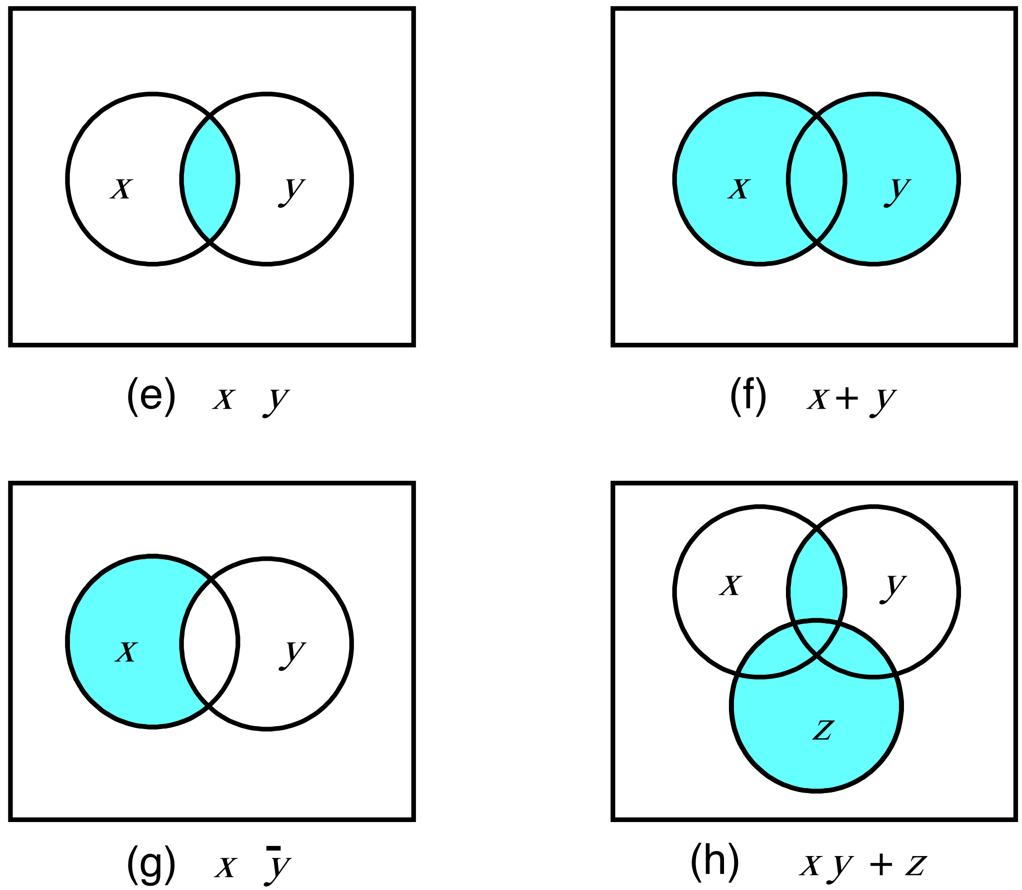
(b) Constant 0

(c) Variable x

(d) $\bar{x}$

[Figure 2.14 from the textbook]

Venn Diagram Basics



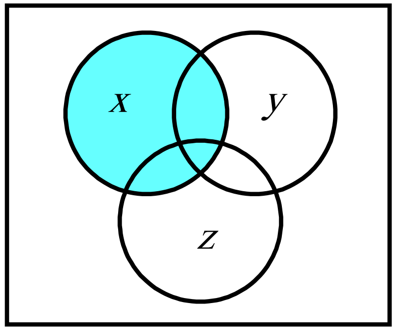
[Figure 2.14 from the textbook]

Let's Prove the Distributive Properties

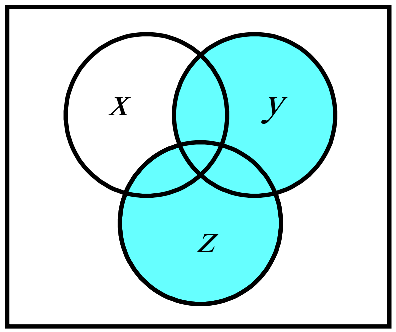
12a. $x \bullet (y + z) = x \bullet y + x \bullet z$

12b. $x + y \bullet z = (x + y) \bullet (x + z)$

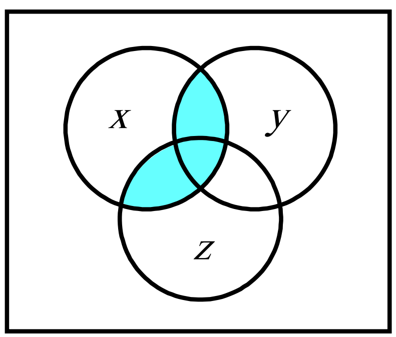
12a. $x \cdot (y + z) = x \cdot y + x \cdot z$



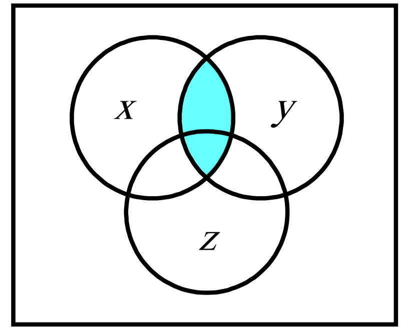
(a) x



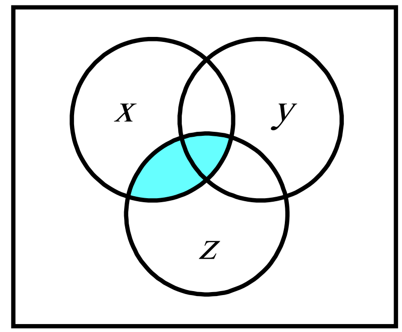
(b) $y + z$



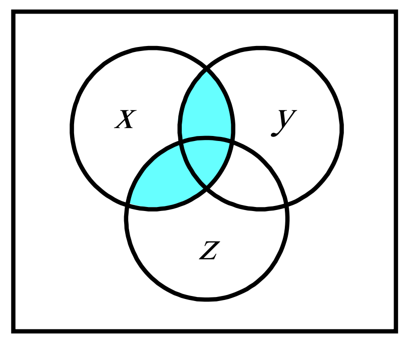
(c) $x \ (y + z)$



(d) $x\,y$



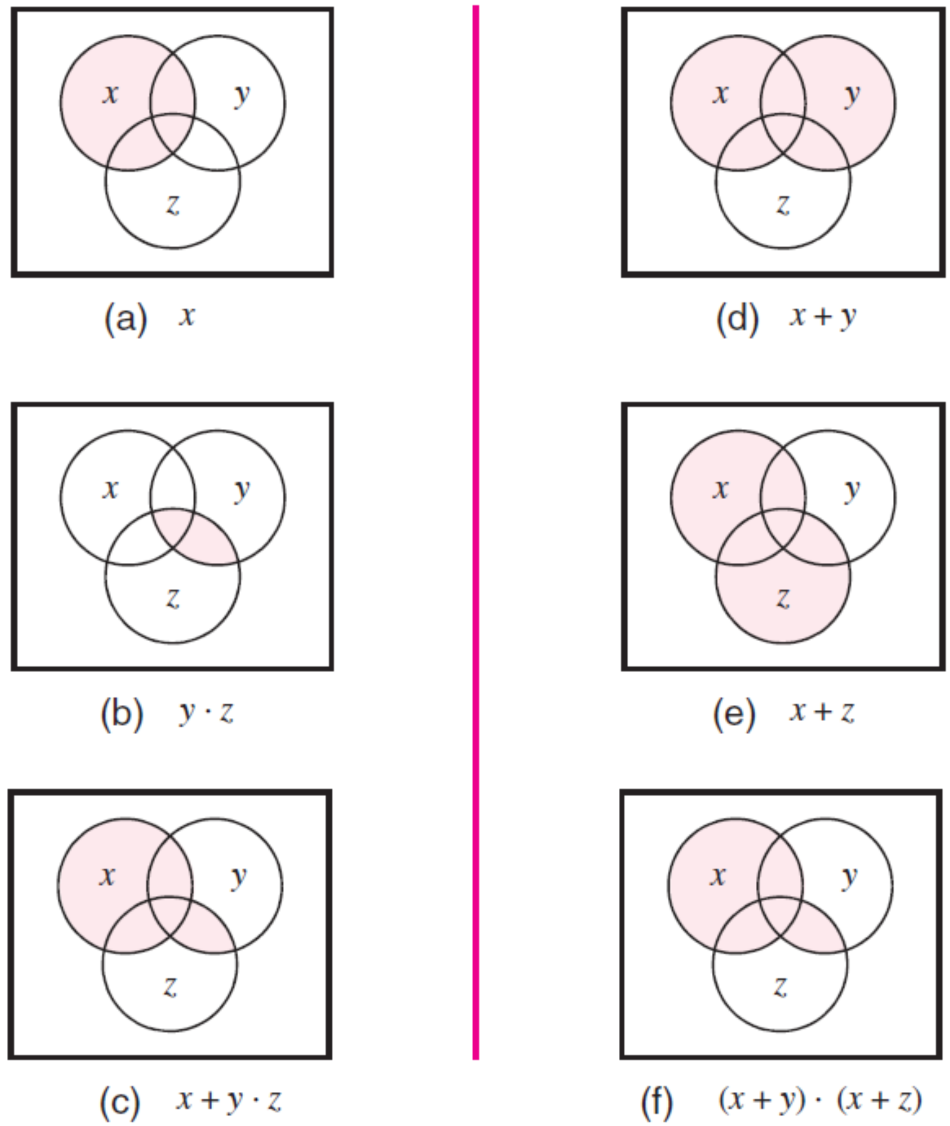
(e) $x\,z$



(f) $x y + x z$

[Figure 2.15 from the textbook]

12b. $x + y \cdot z = (x + y) \cdot (x + z)$



[Figure 2.17 from the textbook]

Try to prove these ones at home

13a. $x + x \bullet y = x$

13b. $x \bullet (x + y) = x$

14a. $x \bullet y + x \bullet \overline{y} = x$

14b. $(x + y) \bullet (x + \overline{y}) = x$

DeMorgan's Theorem

15a. $\overline{x \cdot y} = \overline{x} + \overline{y}$

15b. $\overline{x + y} = \overline{x} \cdot \overline{y}$

Proof of DeMorgan's theorem

15a. $\overline{x \bullet y} = \bar{x} + \bar{y}$

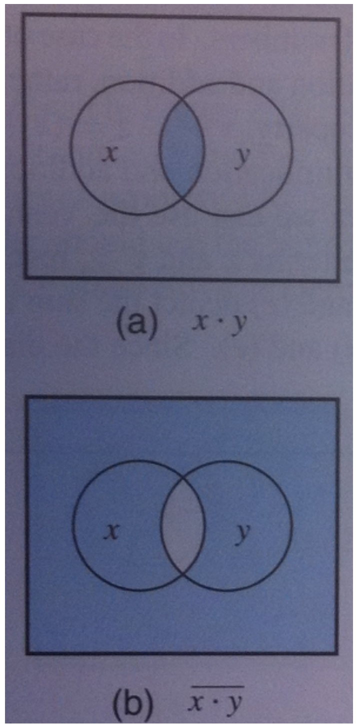
x	y	$x \cdot y$	$\overline{x \cdot y}$	$\bar{x}$	$\bar{y}$	$\bar{x} + \bar{y}$
0	0	0	1	1	1	1
0	1	0	1	1	0	1
1	0	0	1	0	1	1
1	1	1	0	0	0	0

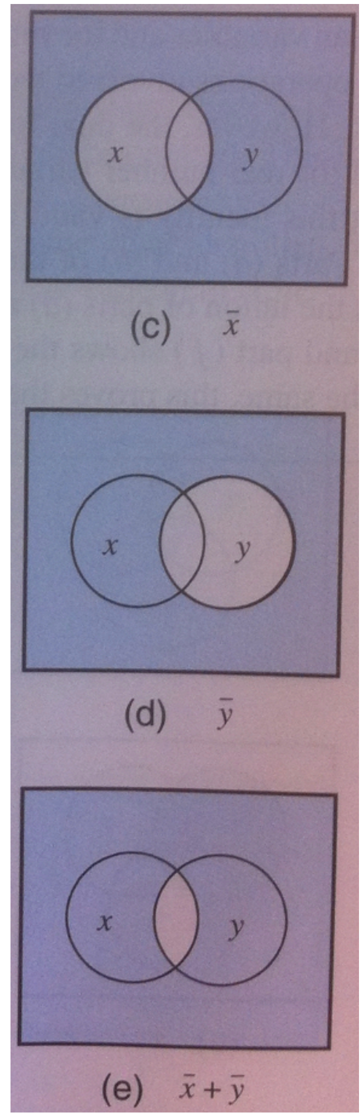
LHS ($x \cdot y$, $\overline{x \cdot y}$)   RHS ($\bar{x}$, $\bar{y}$, $\bar{x} + \bar{y}$)

[Figure 2.13 from the textbook]

Alternative Proof of DeMorgan's theorem

15a. $\overline{x \cdot y} = \overline{x} + \overline{y}$





[Figure 2.18 from the textbook]

Let's prove DeMorgan's theorem

15b. $\overline{x + y} = \bar{x} \bullet \bar{y}$

x	y	$x + y$	$\overline{x + y}$	$\bar{x}$	$\bar{y}$	$\bar{x} \bullet \bar{y}$
0	0					
0	1					
1	0					
1	1					

LHS (columns $x + y$ and $\overline{x + y}$) RHS (columns $\bar{x}$, $\bar{y}$ and $\bar{x} \bullet \bar{y}$)

Try to prove these ones at home

16a. $x + \overline{x} \bullet y = x + y$

16b. $x \bullet (\overline{x} + y) = x \bullet y$

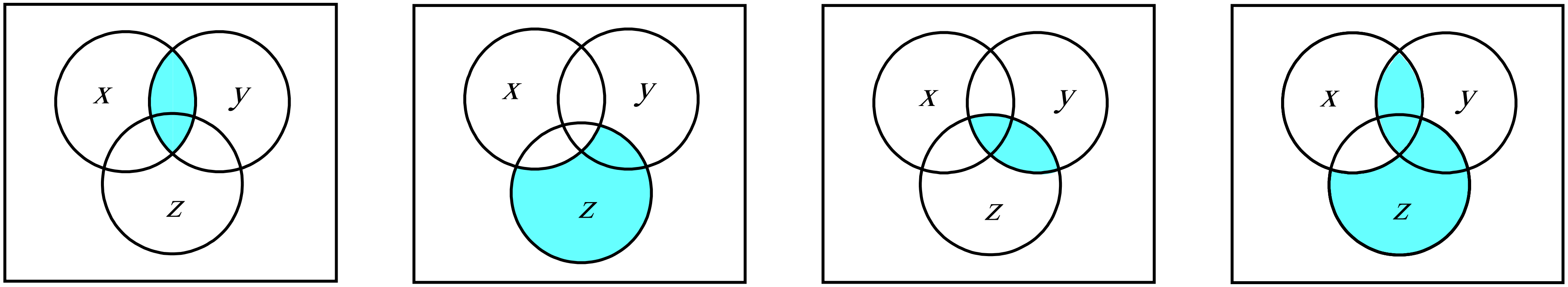
17a. $x \bullet y + y \bullet z + \overline{x} \bullet z = x \bullet y + \overline{x} \bullet z$

17b. $(x+y) \bullet (y+z) \bullet (\overline{x}+z) = (x+y) \bullet (\overline{x}+z)$

Venn Diagram Example
Proof of Property 17a

17a. $x \bullet y + y \bullet z + \overline{x} \bullet z = x \bullet y + \overline{x} \bullet z$

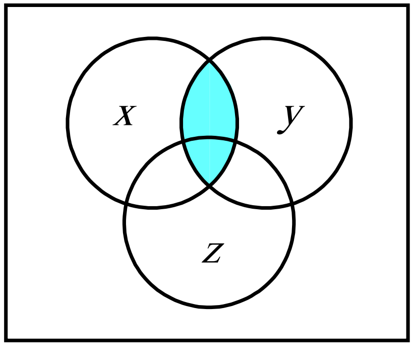
Left-Hand Side



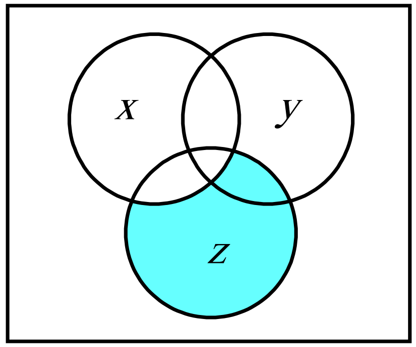
$x\,y$ $\bar{x}\,z$ $y\,z$ $x\,y + \bar{x}\,z + y\,z$

[Figure 2.16 from the textbook]

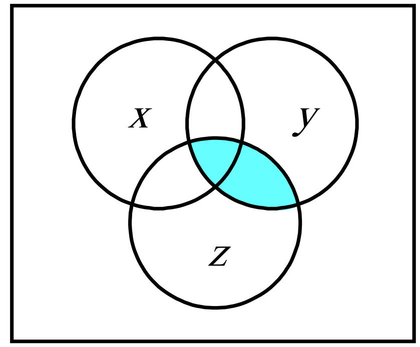
Left-Hand Side



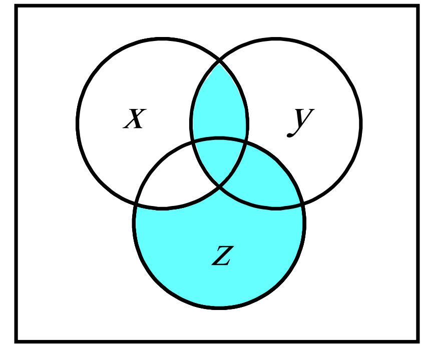
$x y$



$\bar{x} z$

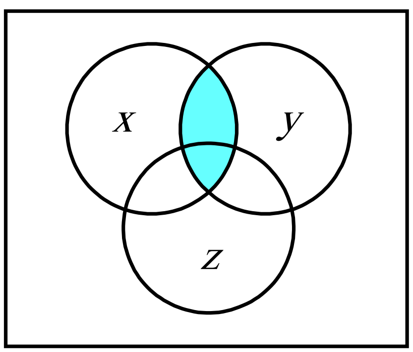


$y z$

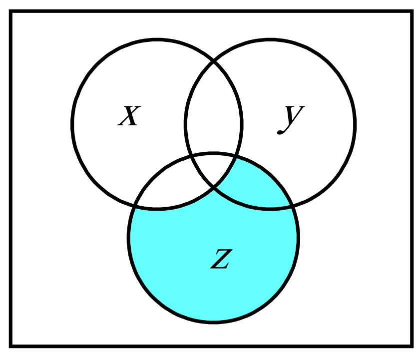


$x y + \bar{x} z + y z$

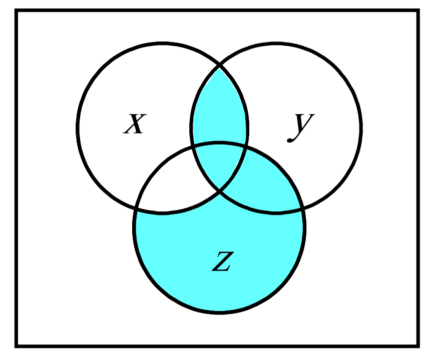
Right-Hand Side



$x y$



$\bar{x} z$

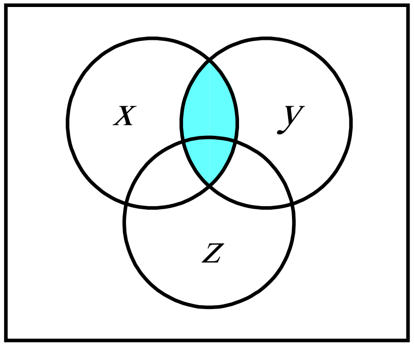


$x y + \bar{x} z$

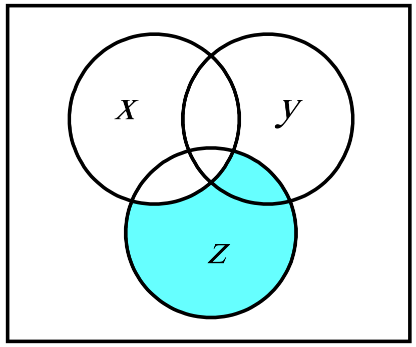
[Figure 2.16 from the textbook]

Left-Hand Side

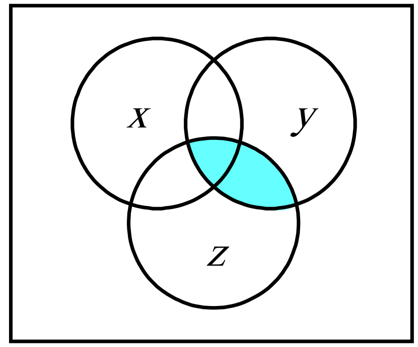
These two are equal



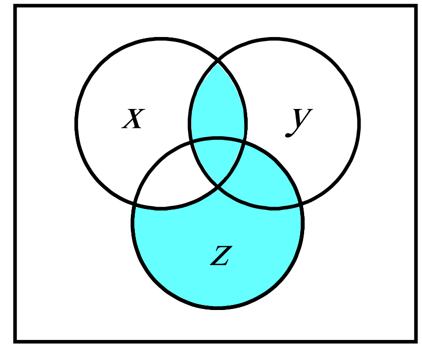
$x y$



$\bar{x} z$

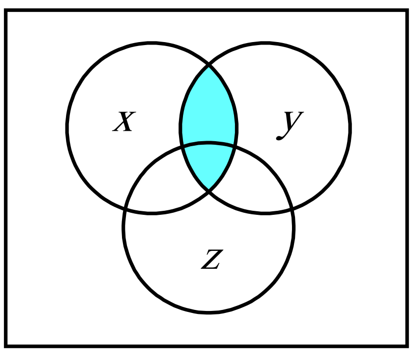


$y z$

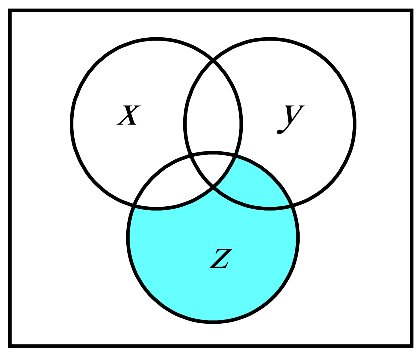


$x y + \bar{x} z + y z$

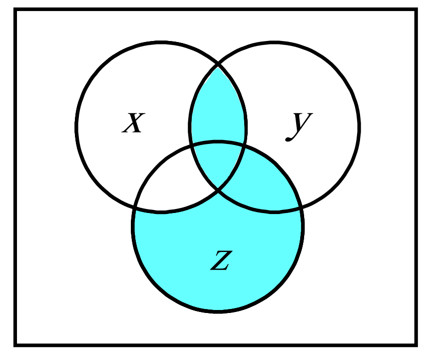
Right-Hand Side



$x y$



$\bar{x} z$



$x y + \bar{x} z$

[Figure 2.16 from the textbook]

Questions?

THE END