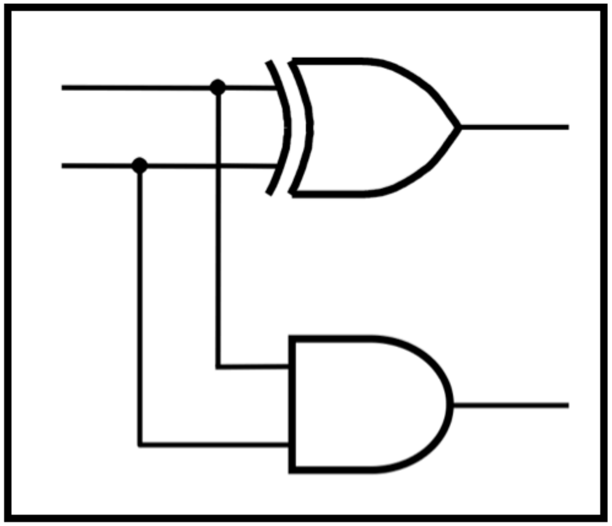


CprE 281: Digital Logic

Instructor: Alexander Stoytchev

http://www.ece.iastate.edu/~alexs/classes/

Signed Numbers

CprE 281: Digital Logic
Iowa State University, Ames, IA

Signed **Integer** Numbers

CprE 281: Digital Logic
Iowa State University, Ames, IA

Quick Review

Adding two bits
(there are four possible cases)

$$\begin{array}{r} x \\ +\,y \\ \hline c\;\,s \end{array} \qquad \begin{array}{r} 0 \\ +\,0 \\ \hline 0\;\,0 \end{array} \qquad \begin{array}{r} 0 \\ +\,1 \\ \hline 0\;\,1 \end{array} \qquad \begin{array}{r} 1 \\ +\,0 \\ \hline 0\;\,1 \end{array} \qquad \begin{array}{r} 1 \\ +\,1 \\ \hline 1\;\,0 \end{array}$$

Carry Sum

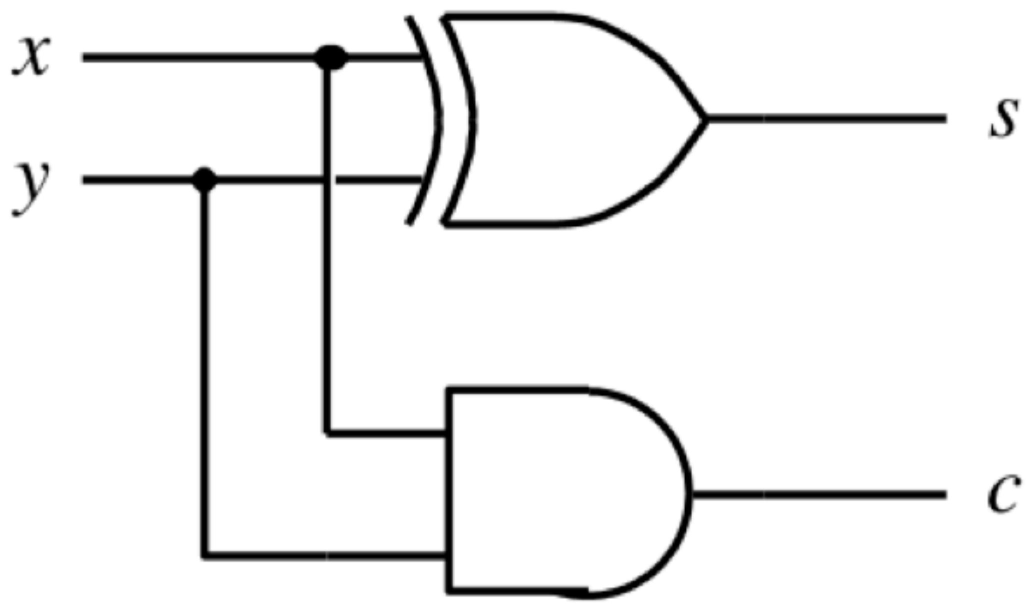
[Figure 3.1a from the textbook]

Adding two bits
(the truth table)

x	y	Carry c	Sum s
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

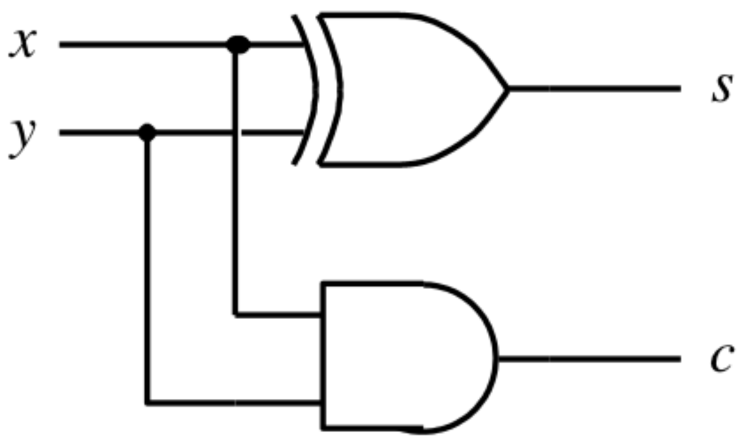
[Figure 3.1b from the textbook]

Adding two bits
(the logic circuit)



[Figure 3.1c from the textbook]

The Half-Adder



(c) Circuit

(d) Graphical symbol

[Figure 3.1c-d from the textbook]

Addition of multibit numbers

$$
\begin{array}{lcccc}
\text{Generated carries} \longrightarrow & & 1\,1\,1\,0 & & \\
X = x_4x_3x_2x_1x_0 & & 0\,1\,1\,1\,1 & & (15)_{10} \\
+\,Y = y_4y_3y_2y_1y_0 & & +\,0\,1\,0\,1\,0 & & +\,(10)_{10} \\
\hline
S = s_4s_3s_2s_1s_0 & & 1\,1\,0\,0\,1 & & (25)_{10}
\end{array}
$$

$$
\begin{array}{ccccc}
\cdots & c_{i+1} & c_i & \cdots \\
\cdots & \cdots & x_i & \cdots \\
\cdots & \cdots & y_i & \cdots \\
\hline
\cdots & \cdots & s_i & \cdots
\end{array}
$$

Bit position i

[Figure 3.2 from the textbook]

Analogy with addition in base 10

$$
\begin{array}{rccc}
 & x_2 & x_1 & x_0 \\
+ & y_2 & y_1 & y_0 \\
\hline
 & s_2 & s_1 & s_0
\end{array}
$$

Analogy with addition in base 10

$$
\begin{array}{r}
3\ 8\ 9 \\
+\quad 1\ 5\ 7 \\
\hline
5\ 4\ 6
\end{array}
$$

Analogy with addition in base 10

carry	0	1	1	0
		3	8	9
+		1	5	7
		5	4	6

Analogy with addition in base 10

$$
\begin{array}{rcccc}
 & c_3 & c_2 & c_1 & c_0 \\
 & & x_2 & x_1 & x_0 \\
+ & & y_2 & y_1 & y_0 \\
\hline
 & & s_2 & s_1 & s_0
\end{array}
$$

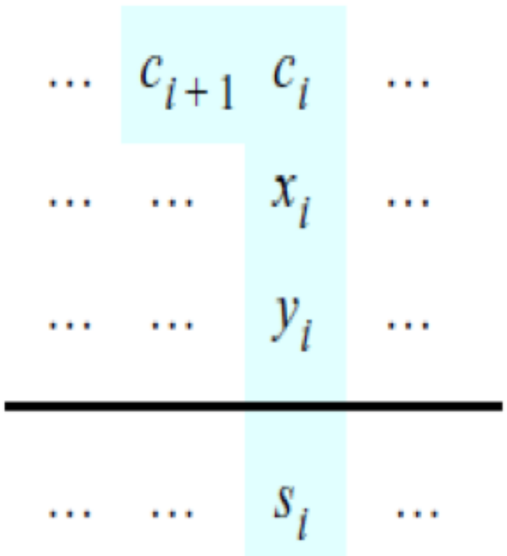
Another example in base 10

$$
\begin{array}{r}
9\;3\;8 \\
+\quad 2\;1\;4 \\
\hline
1\;1\;5\;2
\end{array}
$$

Another example in base 10

carry	1	0	1	0
		9	3	8
+		2	1	4
		1	5	2

Problem Statement and Truth Table



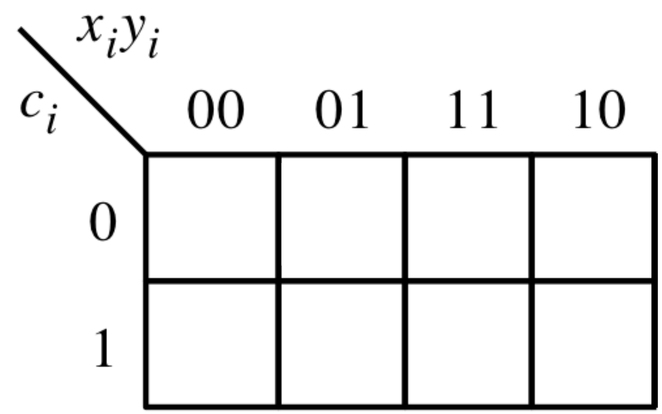
c_i	x_i	y_i	c_{i+1}	s_i
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

[Figure 3.2b from the textbook]

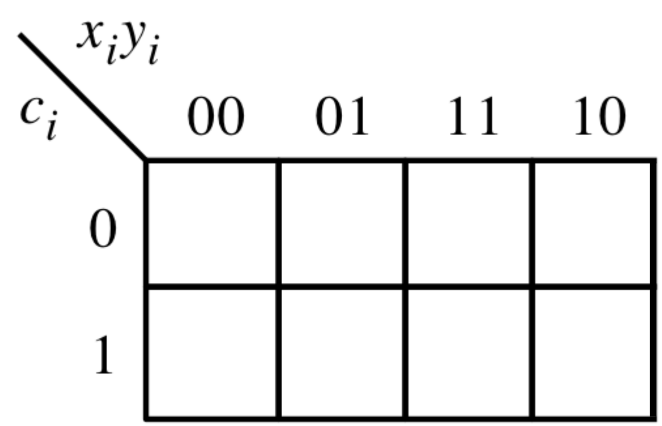
[Figure 3.3a from the textbook]

Let's fill-in the two K-maps

c_i	x_i	y_i	c_{i+1}	s_i
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1



$s_i =$



$c_{i+1} =$

[Figure 3.3a-b from the textbook]

Let’s fill-in the two K-maps

c_i	x_i	y_i	c_{i+1}	s_i
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

c_i \ $x_i y_i$	00	01	11	10
0		1		1
1	1		1	

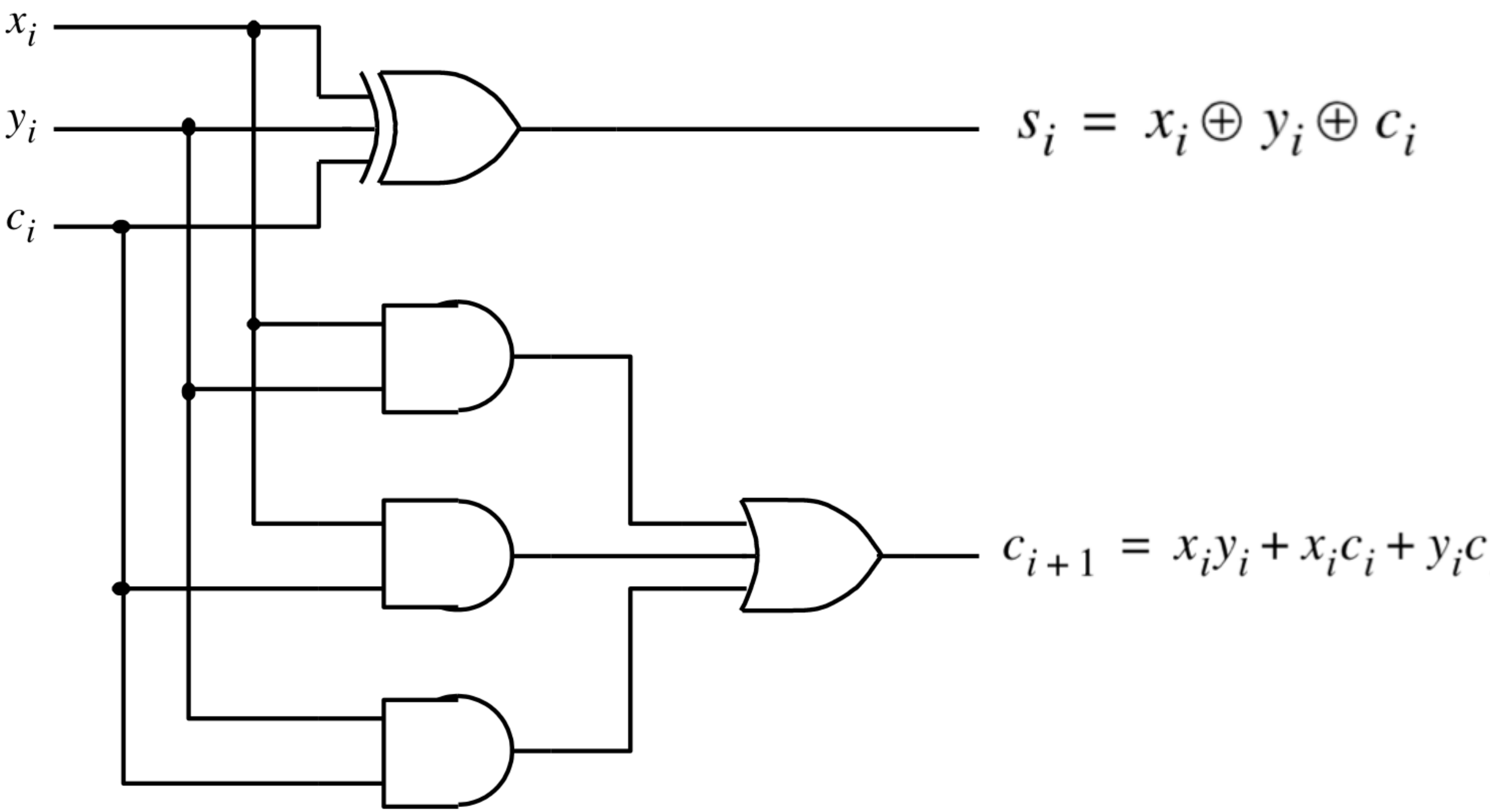
$$s_i = x_i \oplus y_i \oplus c_i$$

c_i \ $x_i y_i$	00	01	11	10
0			1	
1		1	1	1

$$c_{i+1} = x_i y_i + x_i c_i + y_i c_i$$

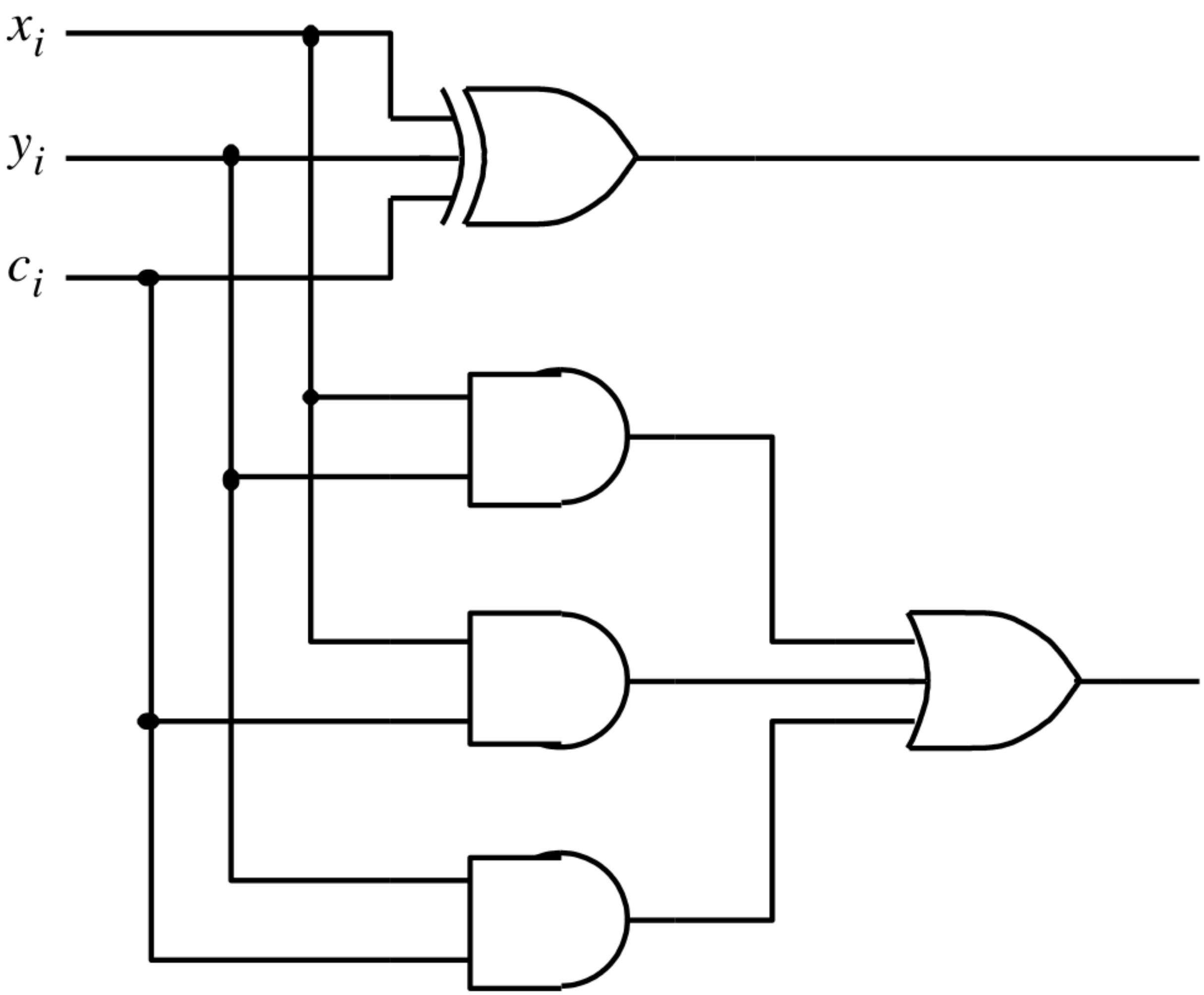
[Figure 3.3a-b from the textbook]

The circuit for the two expressions



[Figure 3.3c from the textbook]

This is called the Full-Adder



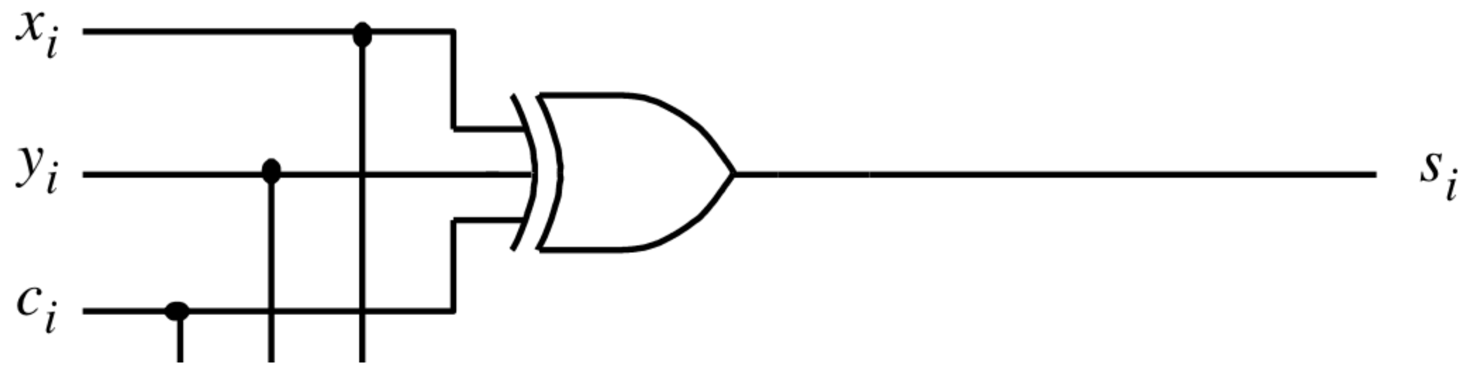
$s_i = x_i \oplus y_i \oplus c_i$

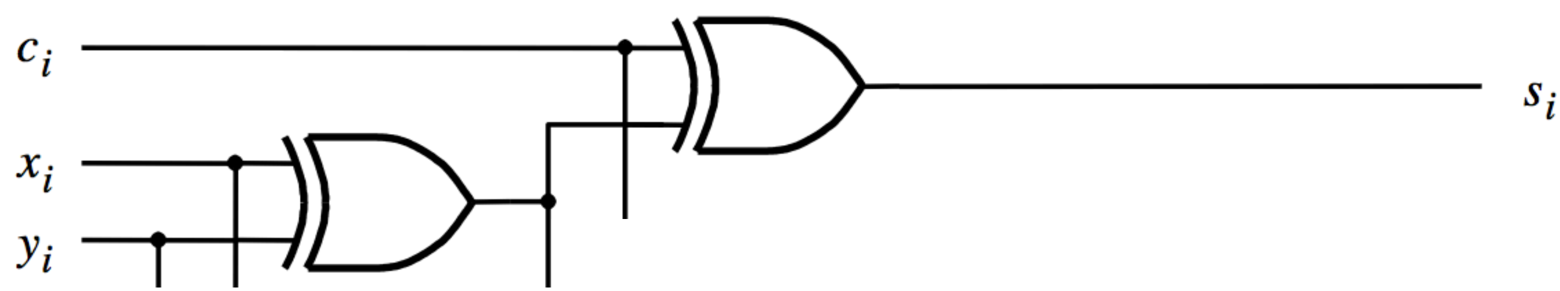
$c_{i+1} = x_i y_i + x_i c_i + y_i c_i$

[Figure 3.3c from the textbook]

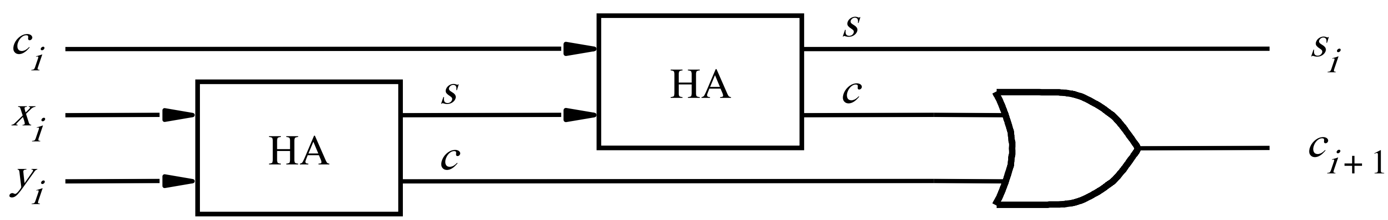
XOR Magic
(s_i can be implemented in two different ways)

$$s_i = x_i \oplus y_i \oplus c_i$$

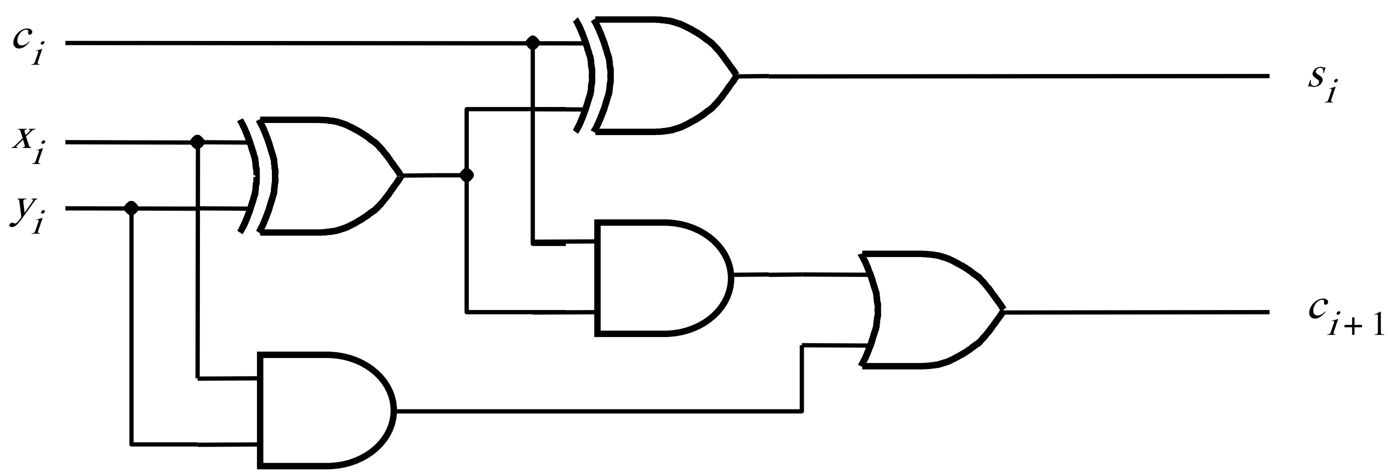




A decomposed implementation of the full-adder circuit



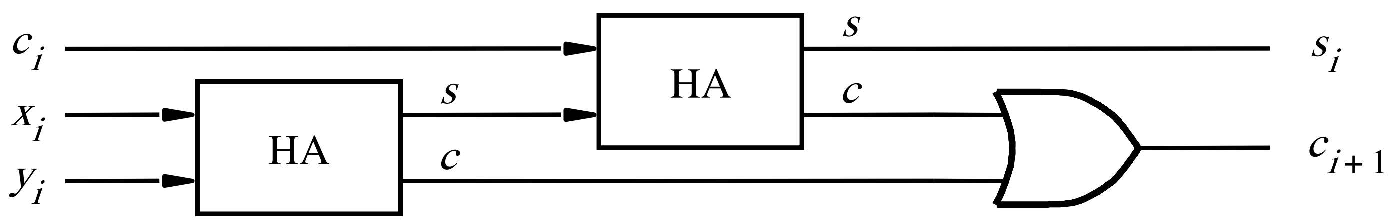
(a) Block diagram



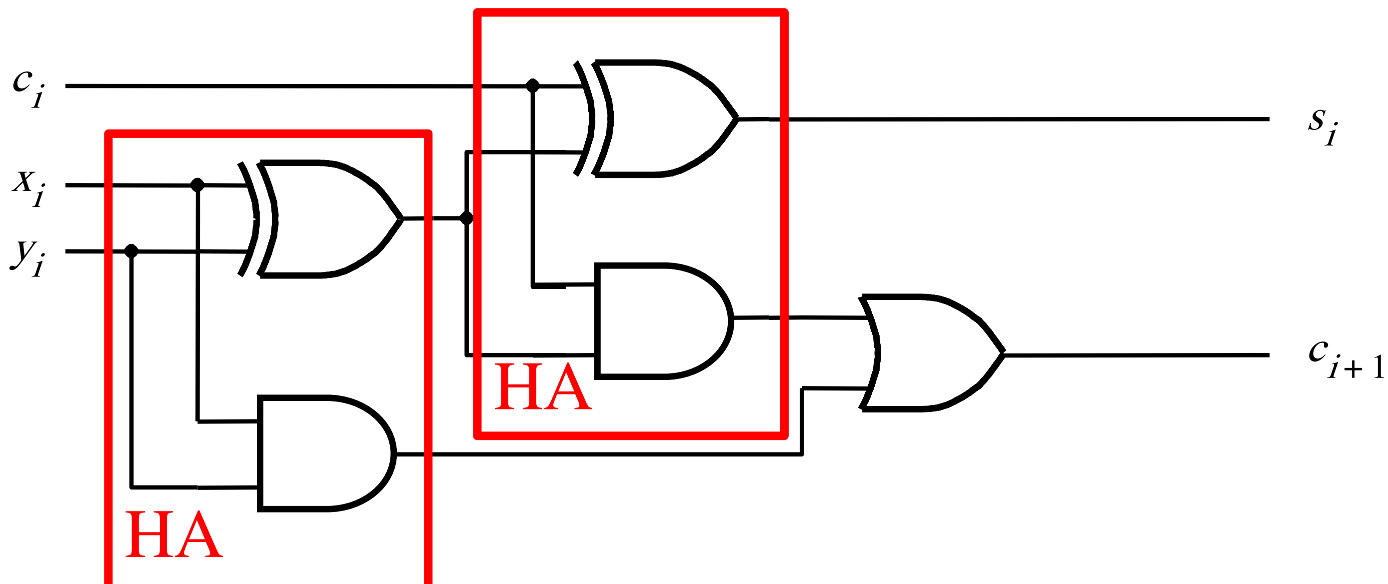
(b) Detailed diagram

[Figure 3.4 from the textbook]

A decomposed implementation of the full-adder circuit



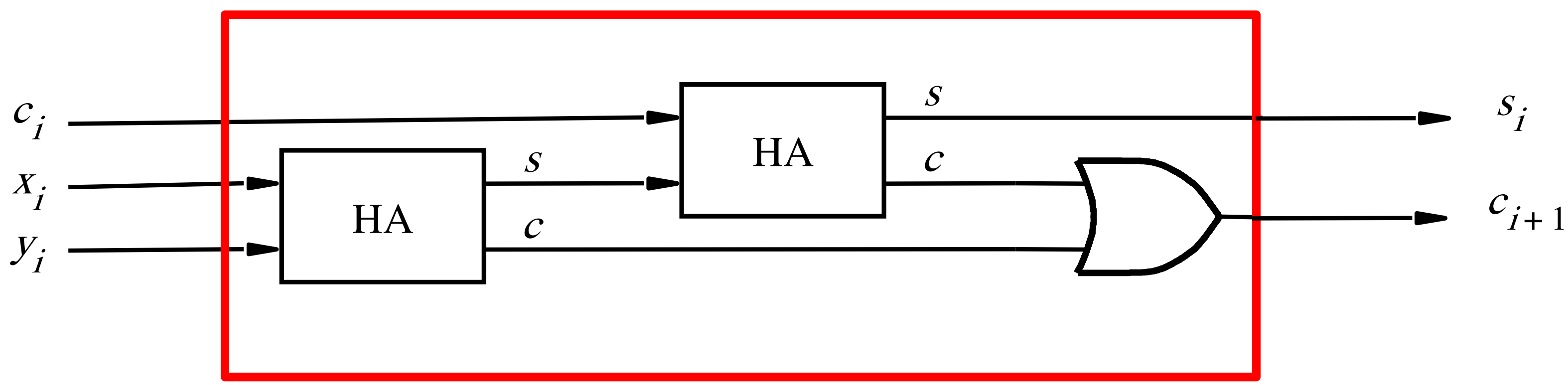
(a) Block diagram



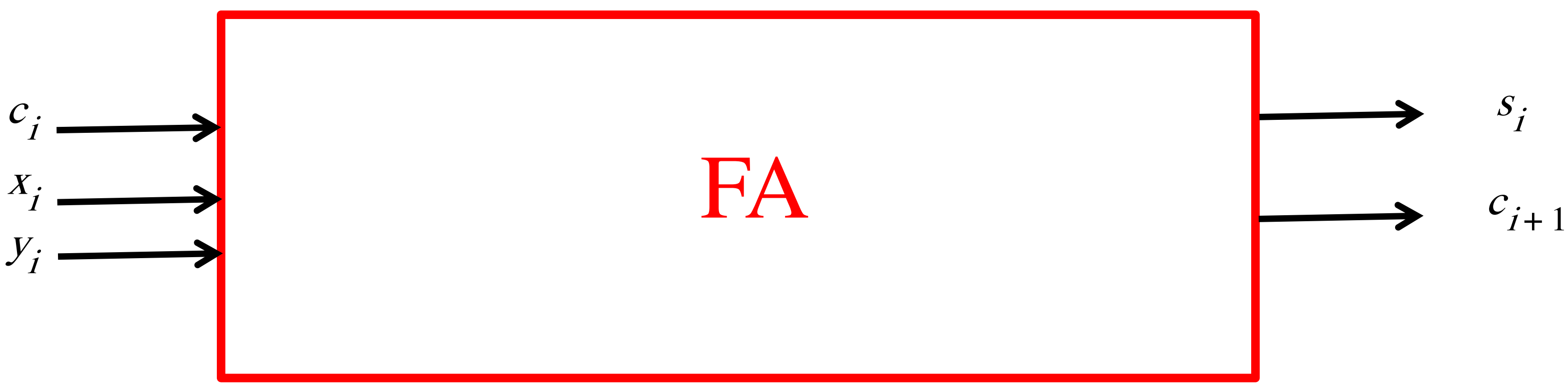
(b) Detailed diagram

[Figure 3.4 from the textbook]

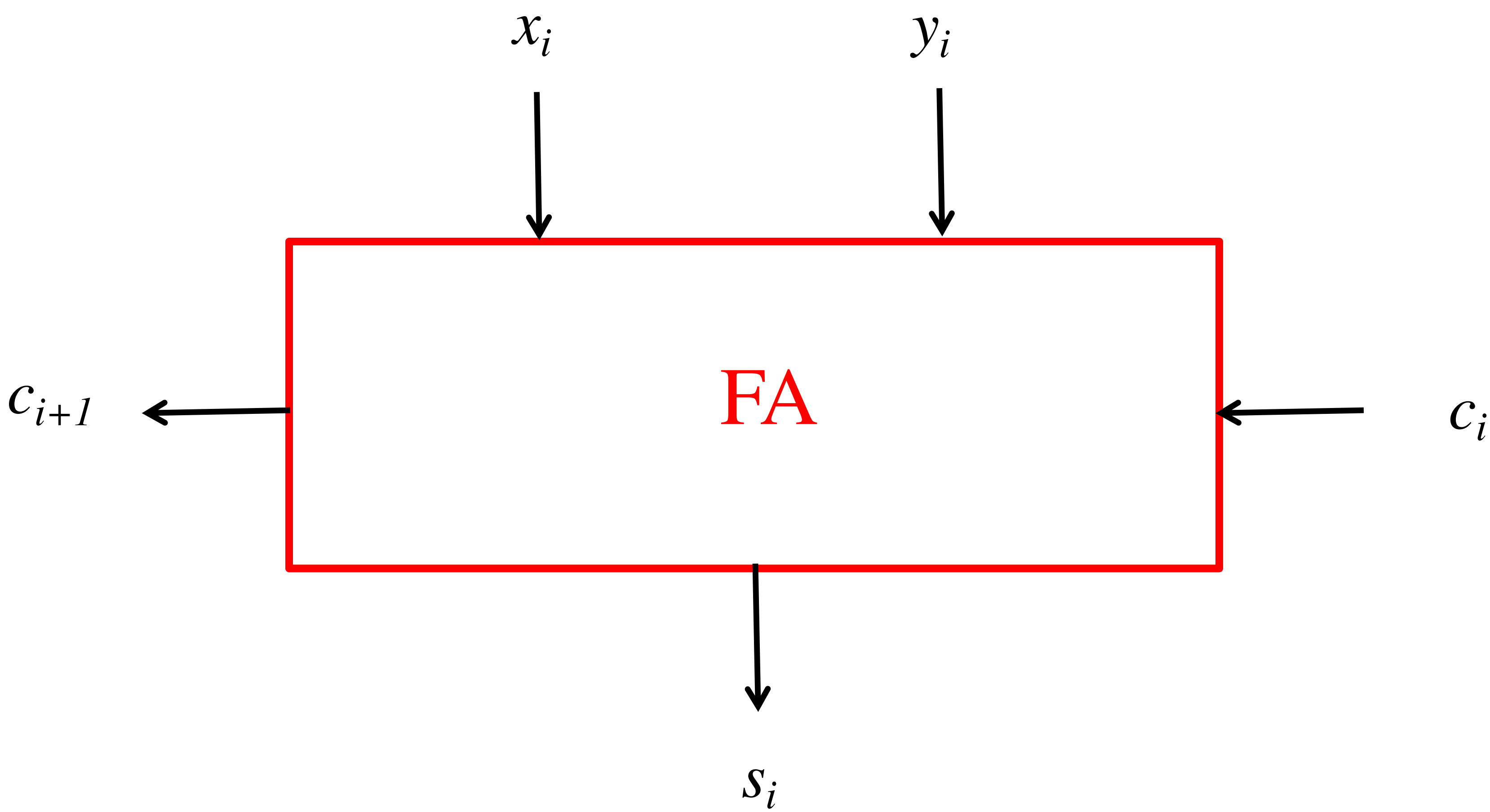
The Full-Adder Abstraction



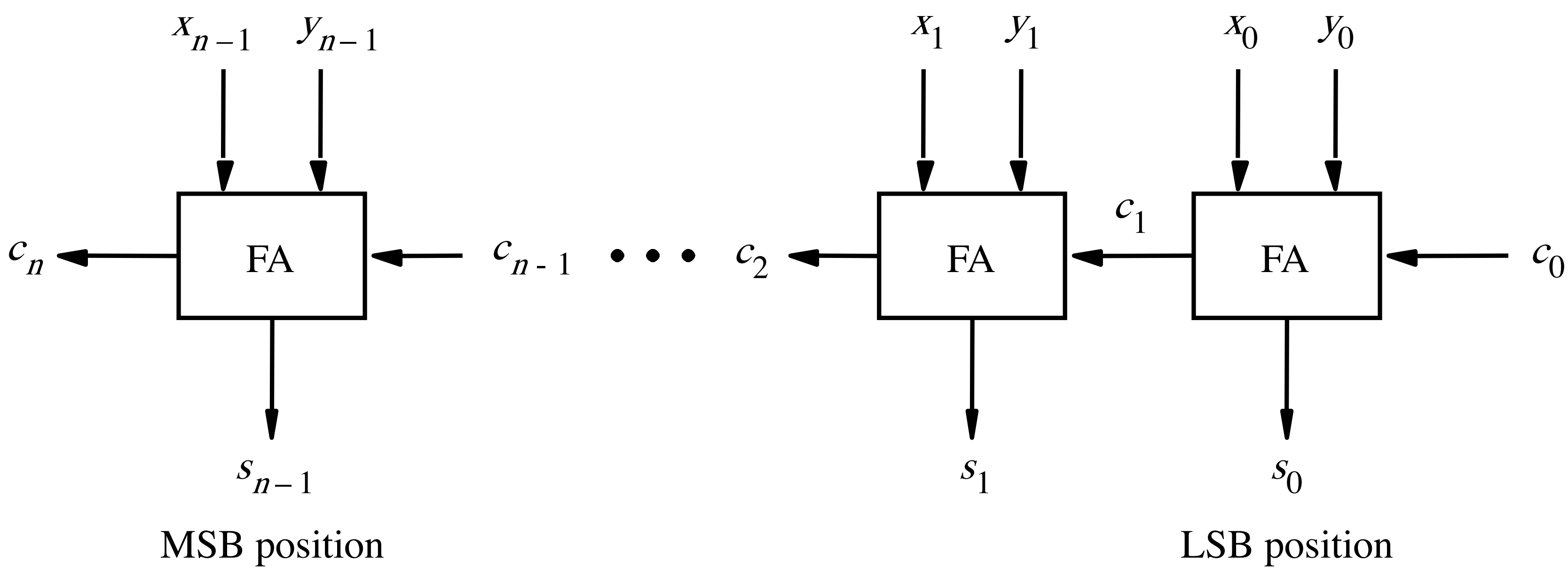
The Full-Adder Abstraction



We can place the arrows anywhere

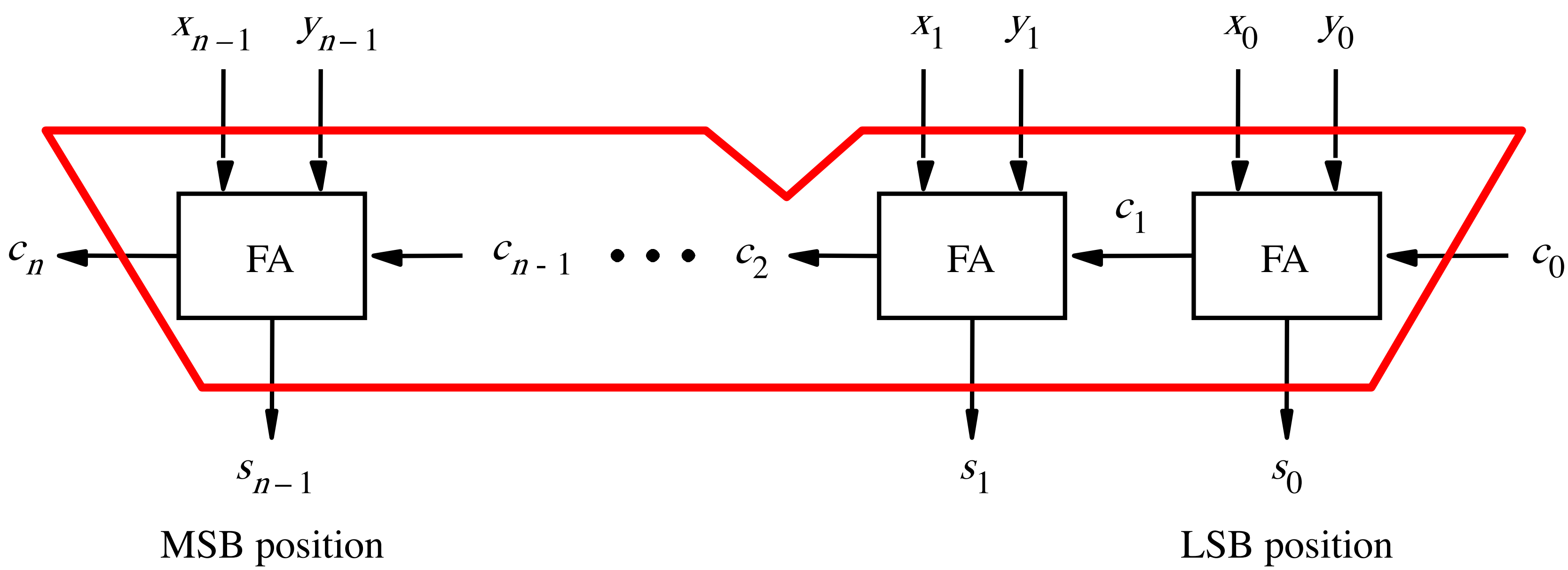


n-bit ripple-carry adder

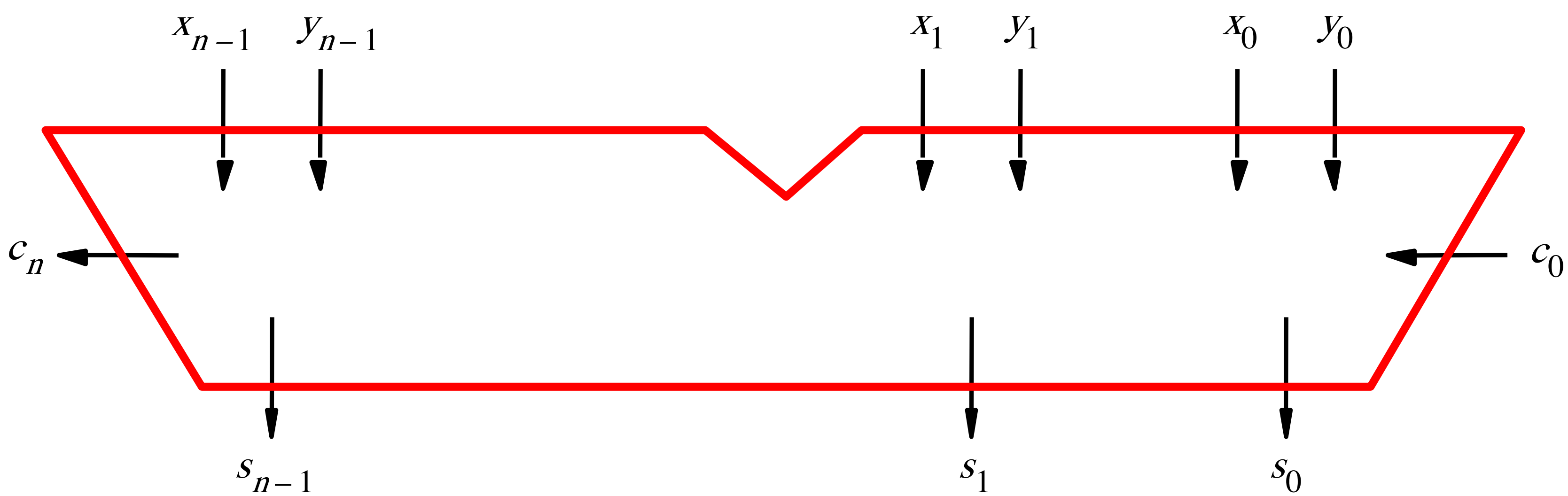


[Figure 3.5 from the textbook]

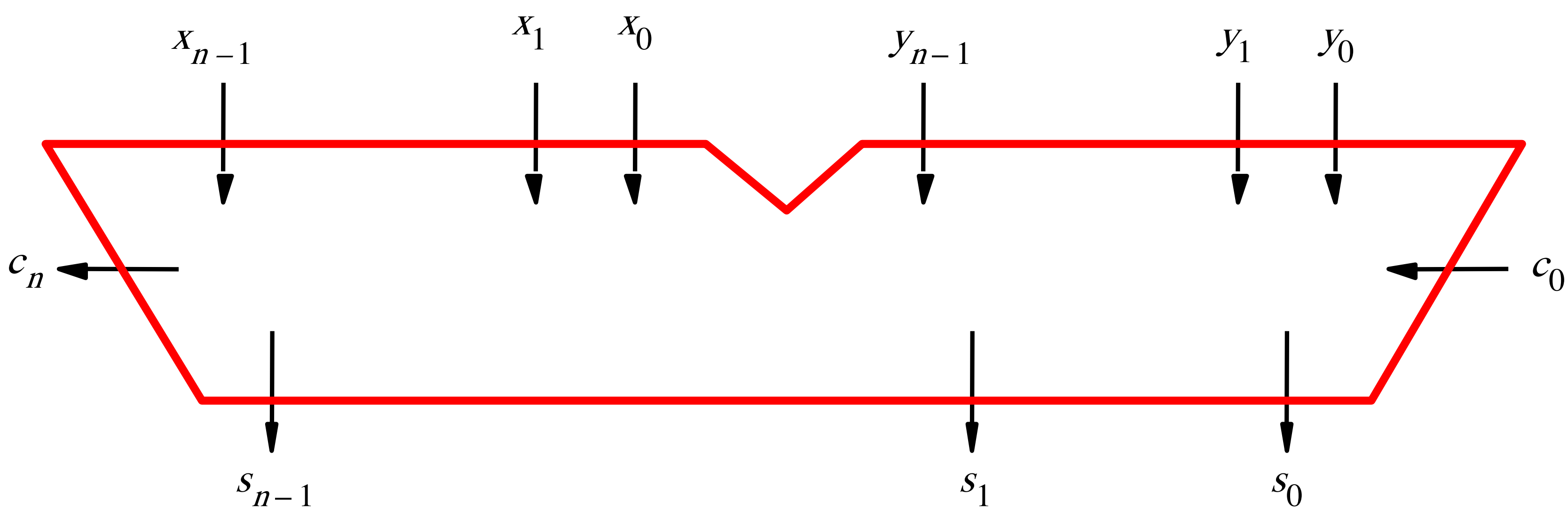
n-bit ripple-carry adder abstraction



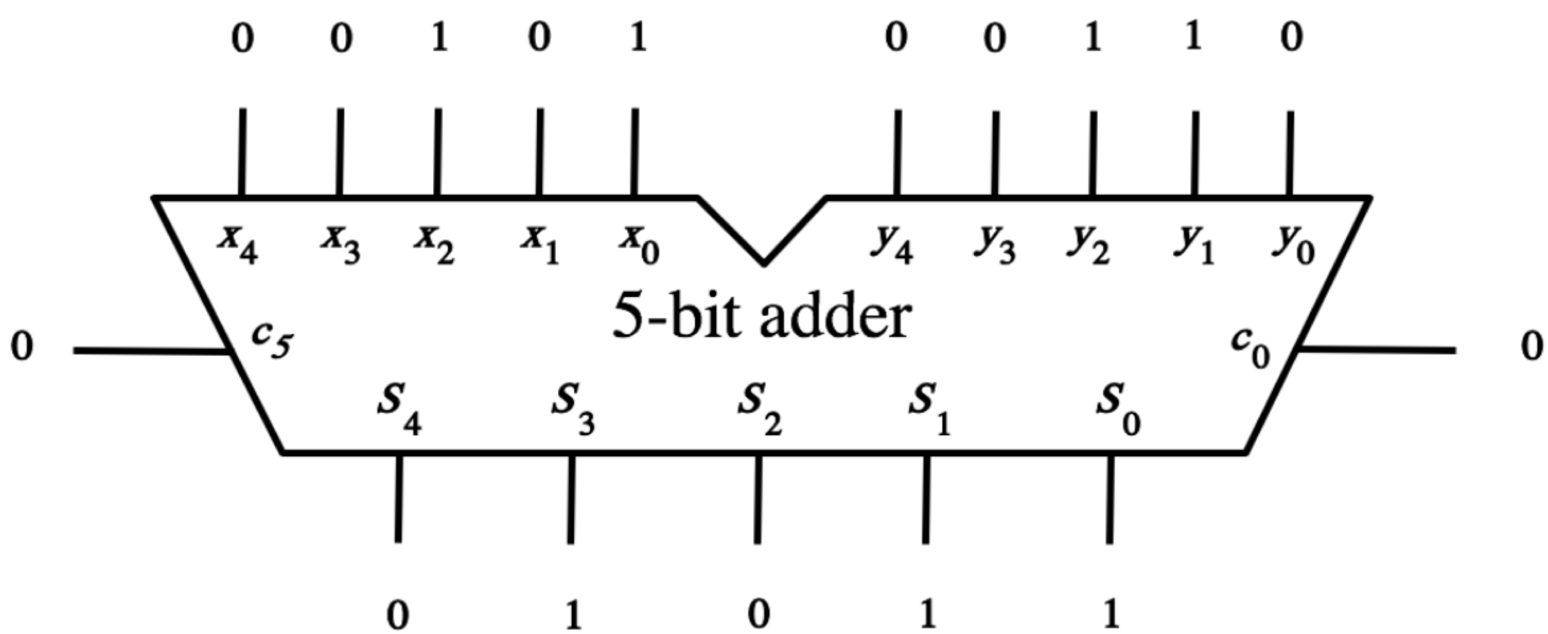
n-bit ripple-carry adder abstraction



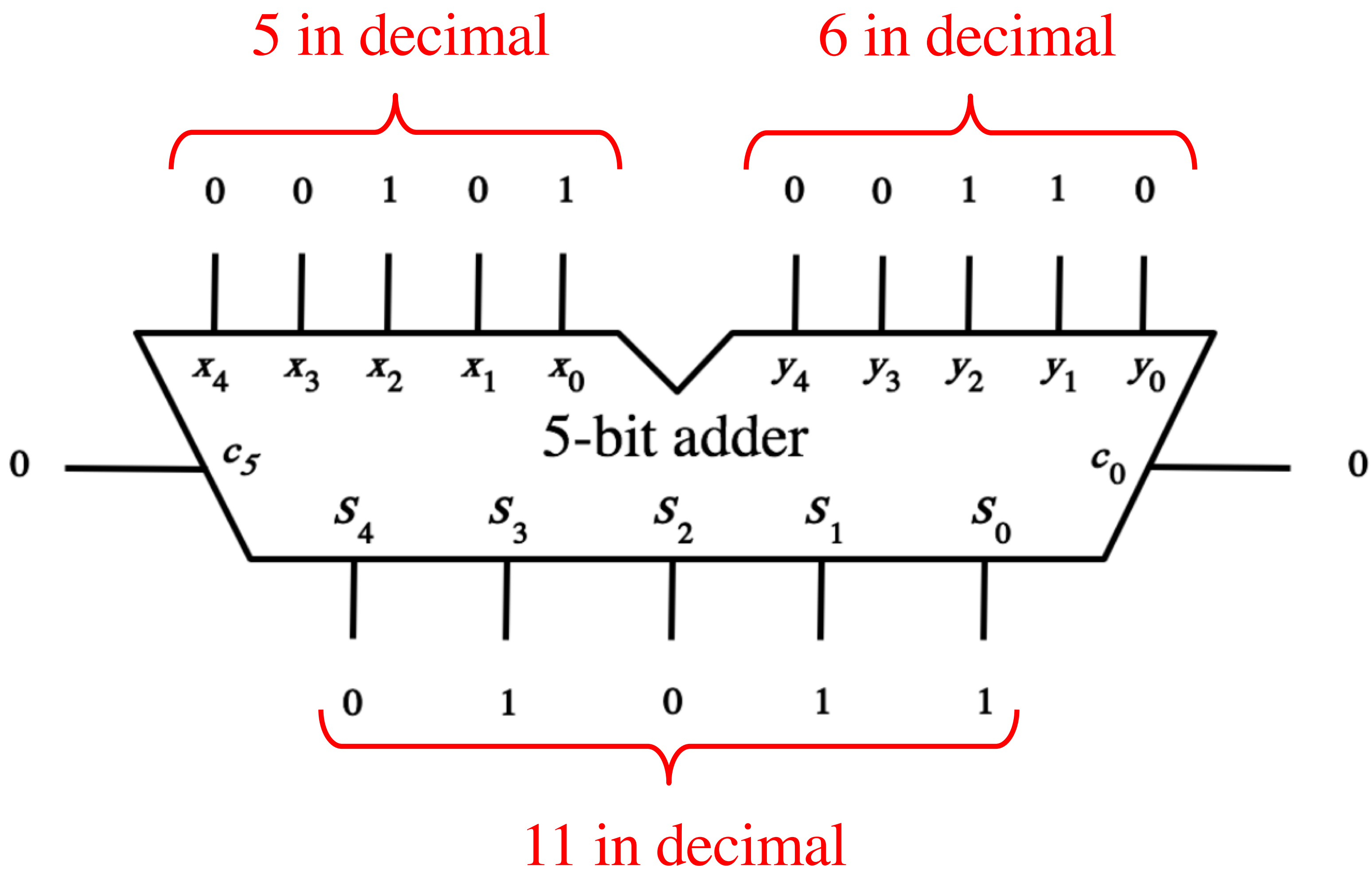
The x and y lines are typically grouped together for better visualization, but the underlying logic remains the same



Example:
Computing 5+6 using a 5-bit adder



Example:
Computing 5+6 using a 5-bit adder



Math Review: Subtraction

$$\begin{array}{r} 39 \\ -\ 15 \\ \hline ?? \end{array}$$

Math Review: Subtraction

$$\begin{array}{r} 39 \\ -\ 15 \\ \hline 24 \end{array}$$

Math Review: Subtraction

$$\begin{array}{r} 82 \\ -\ 61 \\ \hline ?? \end{array} \qquad \begin{array}{r} 48 \\ -\ 26 \\ \hline ?? \end{array} \qquad \begin{array}{r} 32 \\ -\ 11 \\ \hline ?? \end{array}$$

Math Review: Subtraction

$$\begin{array}{r} 82 \\ -\ 61 \\ \hline 21 \end{array} \qquad \begin{array}{r} 48 \\ -\ 26 \\ \hline 22 \end{array} \qquad \begin{array}{r} 32 \\ -\ 11 \\ \hline 21 \end{array}$$

Math Review: Subtraction

$$\begin{array}{r} 82 \\ -\ 64 \\ \hline ?? \end{array} \qquad \begin{array}{r} 48 \\ -\ 29 \\ \hline ?? \end{array} \qquad \begin{array}{r} 32 \\ -\ 13 \\ \hline ?? \end{array}$$

Math Review: Subtraction

$$\begin{array}{r} 82 \\ -\ 64 \\ \hline 18 \end{array} \qquad \begin{array}{r} 48 \\ -\ 29 \\ \hline 19 \end{array} \qquad \begin{array}{r} 32 \\ -\ 13 \\ \hline 19 \end{array}$$

The problems in which row are easier to calculate?

$$\begin{array}{r} 82 \\ -\ 61 \\ \hline ?? \end{array} \qquad \begin{array}{r} 48 \\ -\ 26 \\ \hline ?? \end{array} \qquad \begin{array}{r} 32 \\ -\ 11 \\ \hline ?? \end{array}$$

$$\begin{array}{r} 82 \\ -\ 64 \\ \hline ?? \end{array} \qquad \begin{array}{r} 48 \\ -\ 29 \\ \hline ?? \end{array} \qquad \begin{array}{r} 32 \\ -\ 13 \\ \hline ?? \end{array}$$

The problems in which row are easier to calculate?

$$\begin{array}{r} 82 \\ -\ 61 \\ \hline 21 \end{array} \qquad \begin{array}{r} 48 \\ -\ 26 \\ \hline 22 \end{array} \qquad \begin{array}{r} 32 \\ -\ 11 \\ \hline 21 \end{array}$$

Why?

$$\begin{array}{r} 82 \\ -\ 64 \\ \hline 18 \end{array} \qquad \begin{array}{r} 48 \\ -\ 29 \\ \hline 19 \end{array} \qquad \begin{array}{r} 32 \\ -\ 13 \\ \hline 19 \end{array}$$

Another Way to Do Subtraction

$$82 - 64 = 82 + 100 - 100 - 64$$

Another Way to Do Subtraction

$$82 - 64 = 82 + 100 - 100 - 64$$

$$= 82 + (100 - 64) - 100$$

Another Way to Do Subtraction

$$82 - 64 = 82 + 100 - 100 - 64$$

$$= 82 + (100 - 64) - 100$$

$$= 82 + (99 + 1 - 64) - 100$$

Another Way to Do Subtraction

$$82 – 64 = 82 + 100 – 100 - 64$$

$$= 82 + (100 – 64) - 100$$

$$= 82 + (99 + 1 – 64) - 100$$

$$= 82 + (99 – 64) +1 - 100$$

Another Way to Do Subtraction

$$82 - 64 = 82 + 100 - 100 - 64$$

$$= 82 + (100 - 64) - 100$$

$$= 82 + (99 + 1 - 64) - 100$$

Does not require borrows

$$= 82 + (99 - 64) + 1 - 100$$

9's Complement
(subtract each digit from 9)

$$\begin{array}{r} 99 \\ -\;64 \\ \hline 35 \end{array}$$

10's Complement

(subtract each digit from 9 and add 1 to the result)

$$\begin{array}{r} 99 \\ -\ 64 \\ \hline 35 \end{array} + 1 = 36$$

Another Way to Do Subtraction

$$82 - 64 = 82 + (99 - 64) + 1 - 100$$

Another Way to Do Subtraction

9's complement

$$82 - 64 = 82 + (99 - 64) + 1 - 100$$

Another Way to Do Subtraction

9's complement

$$82 - 64 = 82 + (99 - 64) + 1 - 100$$

$$= 82 + 35 + 1 - 100$$

Another Way to Do Subtraction

9's complement

$$82 - 64 = 82 + (99 - 64) + 1 - 100$$

10's complement

$$= 82 + 35 + 1 - 100$$

Another Way to Do Subtraction

9's complement

$$82 - 64 = 82 + (99 - 64) + 1 - 100$$

10's complement

$$= 82 + 35 + 1 - 100$$

$$= 82 + 36 - 100$$

Another Way to Do Subtraction

9's complement

$$82 - 64 = 82 + (99 - 64) + 1 - 100$$

10's complement

$$= 82 + 35 + 1 - 100$$

$$= 82 + 36 - 100$$ // Add the first two.

$$= 118 - 100$$

Another Way to Do Subtraction

9's complement

$$82 - 64 = 82 + (99 - 64) + 1 - 100$$

10's complement

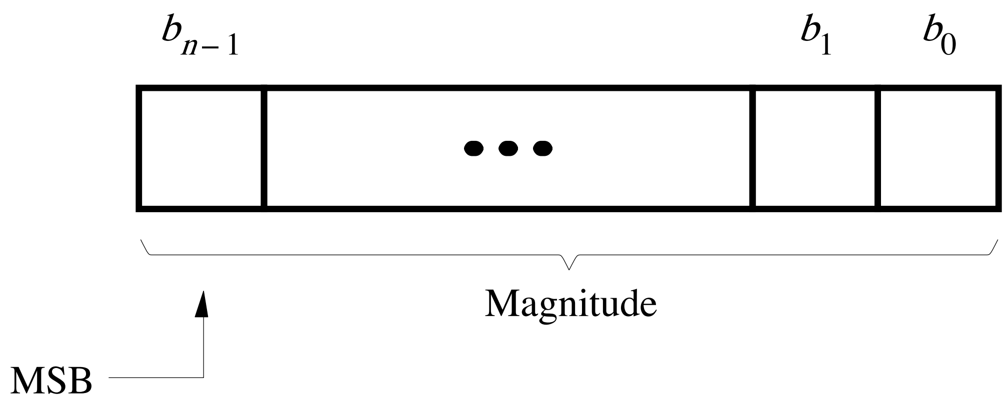
$$= 82 + 35 + 1 - 100$$

$$= 82 + 36 - 100$$ // Add the first two.

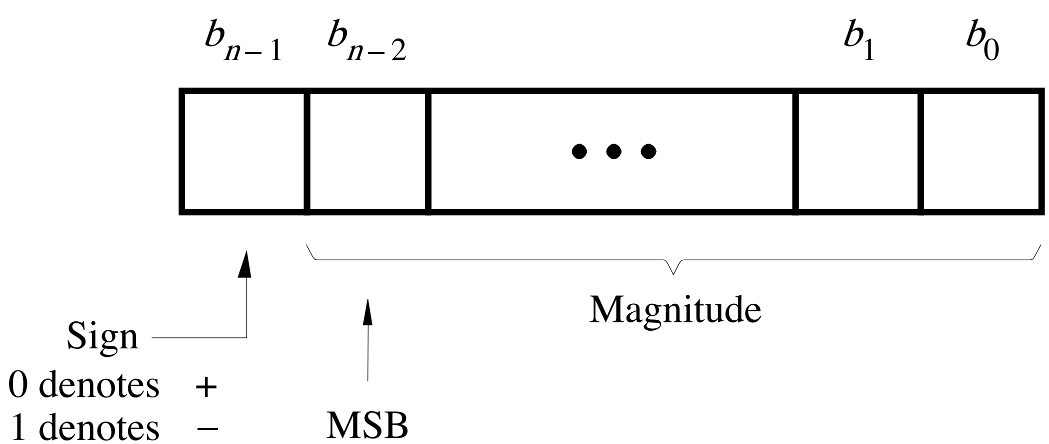
$$= 118 - 100$$ // Just delete the leading 1.
// No need to subtract 100.

$$= 18$$

Formats for representation of integers



(a) Unsigned number



(b) Signed number

[Figure 3.7 from the textbook]

Unsigned Representation

2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
0	0	1	0	1	1	0	0

This represents + 44.

Unsigned Representation

2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
1	0	1	0	1	1	0	0

This represents + 172.

Three Different Ways to Represent Negative Integer Numbers

- Sign and magnitude
- 1’s complement
- 2’s complement

Three Different Ways to Represent Negative Integer Numbers

- Sign and magnitude
- 1's complement
- $\boxed{\text{2's complement}}$

only this method is used in modern computers

Interpretation of four-bit signed integers

$b_3b_2b_1b_0$	Sign and magnitude	1's complement	2's complement
0111	+7	+7	+7
0110	+6	+6	+6
0101	+5	+5	+5
0100	+4	+4	+4
0011	+3	+3	+3
0010	+2	+2	+2
0001	+1	+1	+1
0000	+0	+0	+0
1000	−0	−7	−8
1001	−1	−6	−7
1010	−2	−5	−6
1011	−3	−4	−5
1100	−4	−3	−4
1101	−5	−2	−3
1110	−6	−1	−2
1111	−7	−0	−1

[Table 3.2 from the textbook]

Interpretation of four-bit signed integers

$b_3b_2b_1b_0$	Sign and magnitude	1's complement	2's complement
0111	+7	+7	+7
0110	+6	+6	+6
0101	+5	+5	+5
0100	+4	+4	+4
0011	+3	+3	+3
0010	+2	+2	+2
0001	+1	+1	+1
0000	+0	+0	+0
1000	−0	−7	−8
1001	−1	−6	−7
1010	−2	−5	−6
1011	−3	−4	−5
1100	−4	−3	−4
1101	−5	−2	−3
1110	−6	−1	−2
1111	−7	−0	−1

The top half is the same in all three representations.
It corresponds to the positive integers.

Interpretation of four-bit signed integers

$b_3 b_2 b_1 b_0$	Sign and magnitude	1's complement	2's complement
0111	+7	+7	+7
0110	+6	+6	+6
0101	+5	+5	+5
0100	+4	+4	+4
0011	+3	+3	+3
0010	+2	+2	+2
0001	+1	+1	+1
0000	+0	+0	+0
1000	−0	−7	−8
1001	−1	−6	−7
1010	−2	−5	−6
1011	−3	−4	−5
1100	−4	−3	−4
1101	−5	−2	−3
1110	−6	−1	−2
1111	−7	−0	−1

In all three representations the first bit represents the sign.
If that bit is 1, then the number is negative.

Interpretation of four-bit signed integers

$b_3b_2b_1b_0$	Sign and magnitude	1's complement	2's complement
0111	+7	+7	+7
0110	+6	+6	+6
0101	+5	+5	+5
0100	+4	+4	+4
0011	+3	+3	+3
0010	+2	+2	+2
0001	+1	+1	+1
0000	+0	+0	+0
1000	−0	−7	−8
1001	−1	−6	−7
1010	−2	−5	−6
1011	−3	−4	−5
1100	−4	−3	−4
1101	−5	−2	−3
1110	−6	−1	−2
1111	−7	−0	−1

Notice that in this representation there are two zeros!

Interpretation of four-bit signed integers

$b_3b_2b_1b_0$	Sign and magnitude	1's complement	2's complement
0111	$+7$	$+7$	$+7$
0110	$+6$	$+6$	$+6$
0101	$+5$	$+5$	$+5$
0100	$+4$	$+4$	$+4$
0011	$+3$	$+3$	$+3$
0010	$+2$	$+2$	$+2$
0001	$+1$	$+1$	$+1$
0000	$+0$	$+0$	$+0$
1000	-0	-7	-8
1001	-1	-6	-7
1010	-2	-5	-6
1011	-3	-4	-5
1100	-4	-3	-4
1101	-5	-2	-3
1110	-6	-1	-2
1111	-7	-0	-1

There are two zeros in this representation as well!

Interpretation of four-bit signed integers

$b_3b_2b_1b_0$	Sign and magnitude	1's complement	2's complement
0111	$+7$	$+7$	$+7$
0110	$+6$	$+6$	$+6$
0101	$+5$	$+5$	$+5$
0100	$+4$	$+4$	$+4$
0011	$+3$	$+3$	$+3$
0010	$+2$	$+2$	$+2$
0001	$+1$	$+1$	$+1$
0000	$+0$	$+0$	$+0$
1000	-0	-7	-8
1001	-1	-6	-7
1010	-2	-5	-6
1011	-3	-4	-5
1100	-4	-3	-4
1101	-5	-2	-3
1110	-6	-1	-2
1111	-7	-0	-1

In this representation there is one more negative number.

Sign and Magnitude

Sign and Magnitude Representation
(using the left-most bit as the sign)

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0
0	0	1	0	1	1	0	0

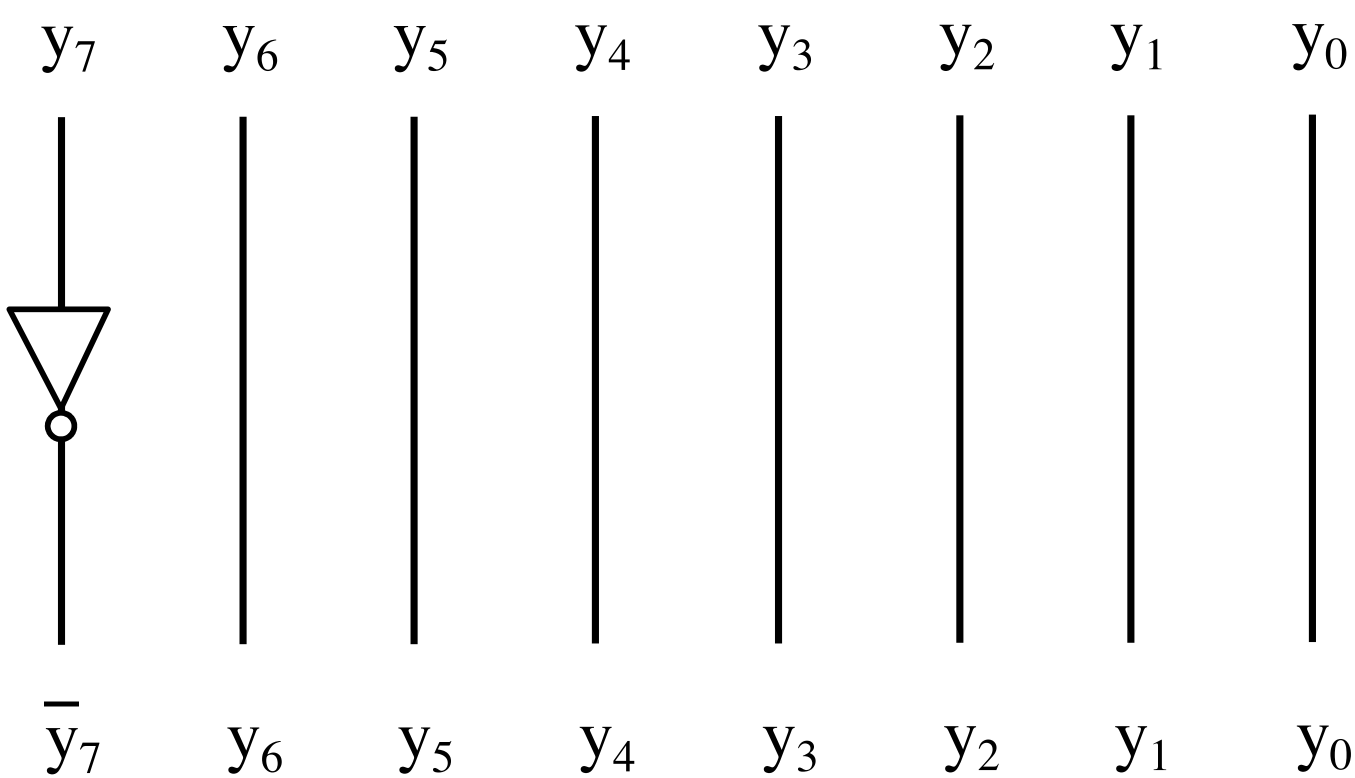
This represents + 44.

Sign and Magnitude Representation
(using the left-most bit as the sign)

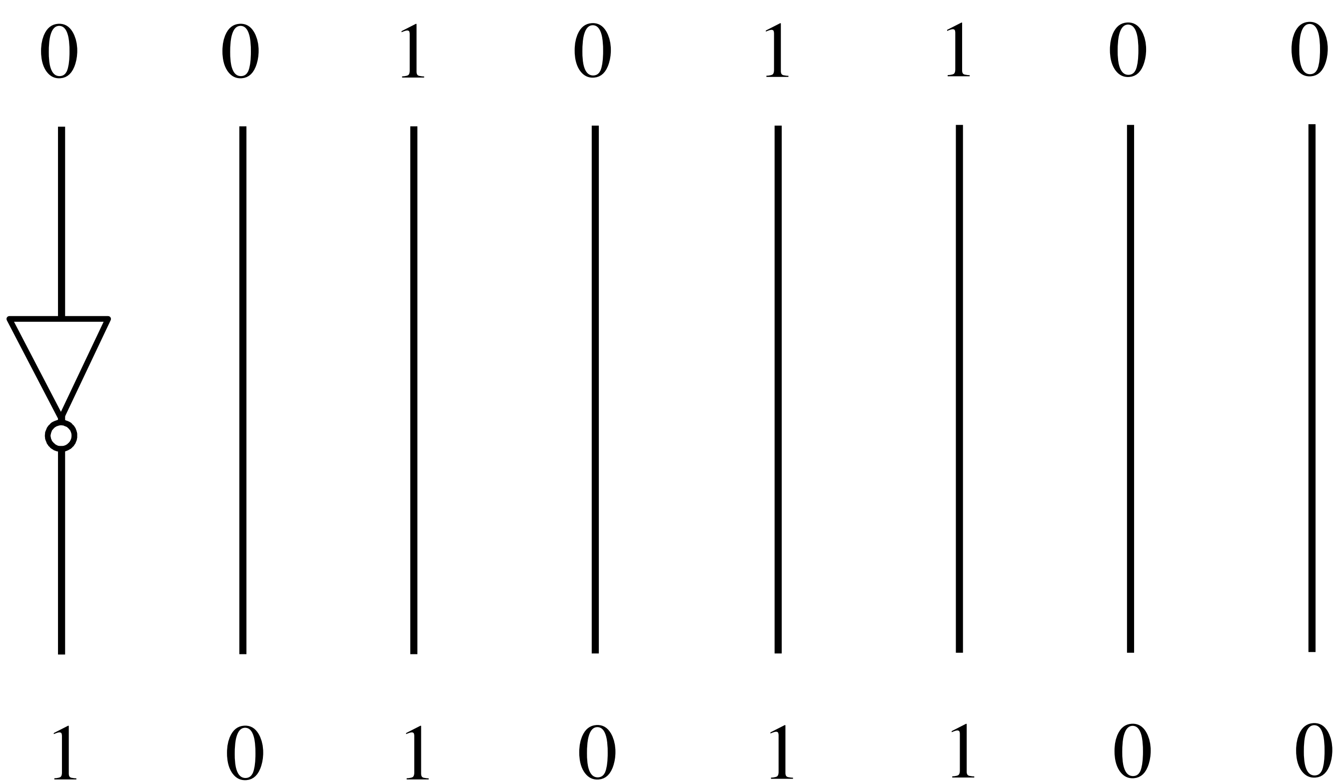
sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0
1	0	1	0	1	1	0	0

This represents – 44.

Circuit for negating a number stored in sign and magnitude representation



Circuit for negating a number stored in sign and magnitude representation



1’s Complement

1’s complement
(subtract each digit from 1)

Let K be the negative equivalent of an n-bit positive number P.

Then, in 1’s complement representation K is obtained by subtracting P from $2^n - 1$, namely

$$K = (2^n - 1) - P$$

This means that K can be obtained by inverting all bits of P.

1's complement
(subtract each digit from 1)

Let K be the negative equivalent of an 8-bit positive number P.

Then, in 1's complement representation K is obtained by subtracting P from $2^8 - 1$, namely

$$K = (2^8 - 1) - P = 255 - P$$

This means that K can be obtained by inverting all bits of P.

Provided that P is between 0 and 127, because the most significant bit must be zero to indicate that it is positive.

1's Complement Representation

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0
0	0	1	0	1	1	0	0

1’s Complement Representation

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0
0	0	1	0	1	1	0	0

+ 44

$$2^5 + 2^3 + 2^2 = 44$$

1's Complement Representation

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
0	0	1	0	1	1	0	0	+ 44

+ 44 in 1's complement representation

1's Complement Representation
(invert all the bits to negate the number)

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
0	0	1	0	1	1	0	0	+ 44

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
1	1	0	1	0	0	1	1	- 44

1's Complement Representation
(invert all the bits to negate the number)

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
0	0	1	0	1	1	0	0	+ 44
positive								

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
1	1	0	1	0	0	1	1	- 44
negative								

1's Complement Representation
(invert all the bits to negate the number)

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
0	0	1	0	1	1	0	0	+ 44

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
1	1	0	1	0	0	1	1	- 44

$2^7 + 2^6 + 2^4 + 2^1 + 2^0 = 211$ (as unsigned)

1's Complement Representation
(invert all the bits to negate the number)

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
0	0	1	0	1	1	0	0	+ 44

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
1	1	0	1	0	0	1	1	- 44

211 = 255 – 44 (as unsigned)

1’s Complement Representation
(invert all the bits to negate the number)

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
0	0	1	0	1	1	0	0	+ 44

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
1	1	0	1	0	0	1	1	- 44

- 44 in 1’s complement representation

1's complement
(subtract each digit from 1)

$$
\begin{array}{rcccccccc}
 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
- & & & & & & & & \\
 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
\hline
 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1
\end{array}
$$

1's complement
(subtract each digit from 1)

No need to borrow!

	1	1	1	1	1	1	1	1
−	0	0	1	0	1	1	0	0
	1	1	0	1	0	0	1	1

1’s complement
(subtract each digit from 1)

$$
\begin{array}{r|cccccccc|l}
 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 255 \\
- & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 44 \\
\hline
 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 &
\end{array}
$$

1's complement
(subtract each digit from 1)

$$
\begin{array}{r}
\;1\;1\;1\;1\;1\;1\;1\;1 \\
-\;0\;0\;1\;0\;1\;1\;0\;0 \\
\hline
\boxed{1\;1\;0\;1\;0\;0\;1\;1}\;\; 211
\end{array}
$$

211 = 255 – 44 (as unsigned)

1's complement
(subtract each digit from 1)

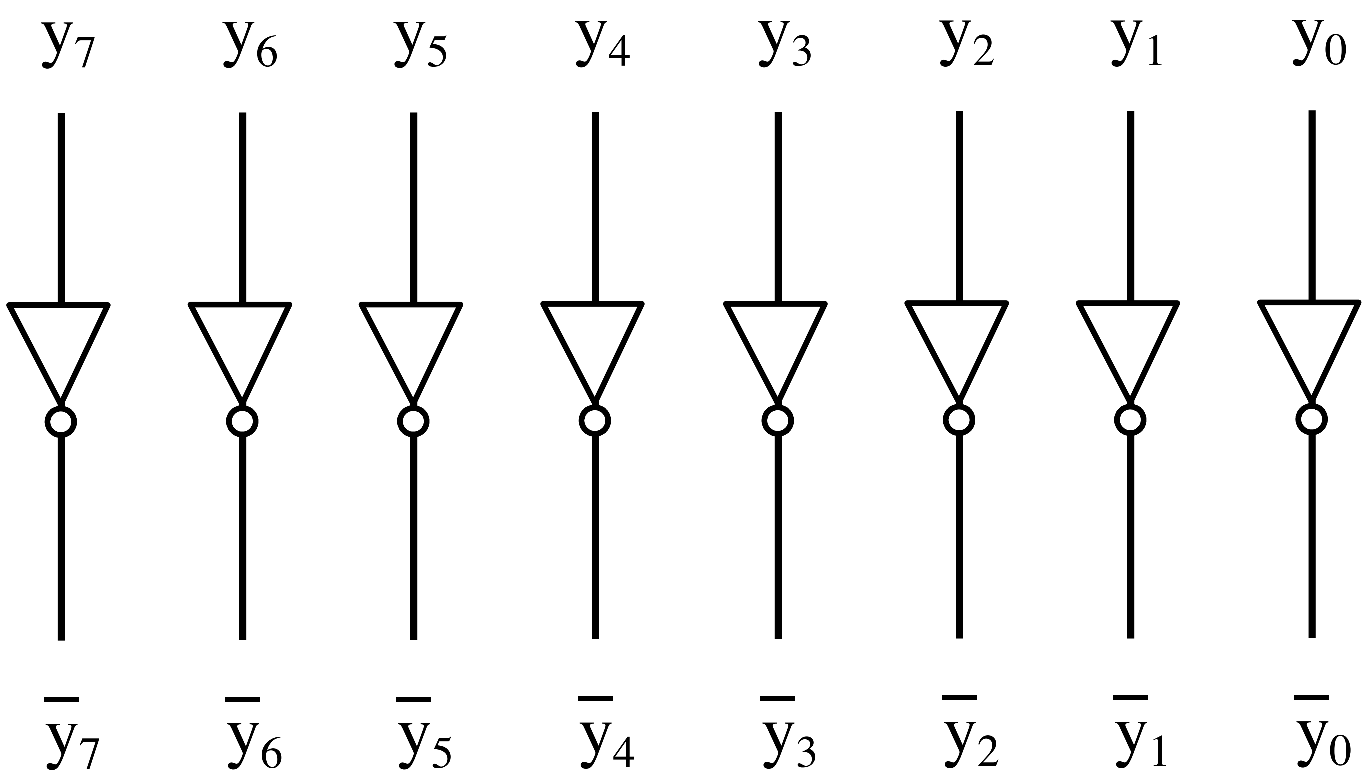
$$
\begin{array}{cccccccccc}
 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \\
- & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & \\
\hline
 & \boxed{1} & \boxed{1} & \boxed{0} & \boxed{1} & \boxed{0} & \boxed{0} & \boxed{1} & \boxed{1} & -44
\end{array}
$$

211 = 255 – 44 (as unsigned)

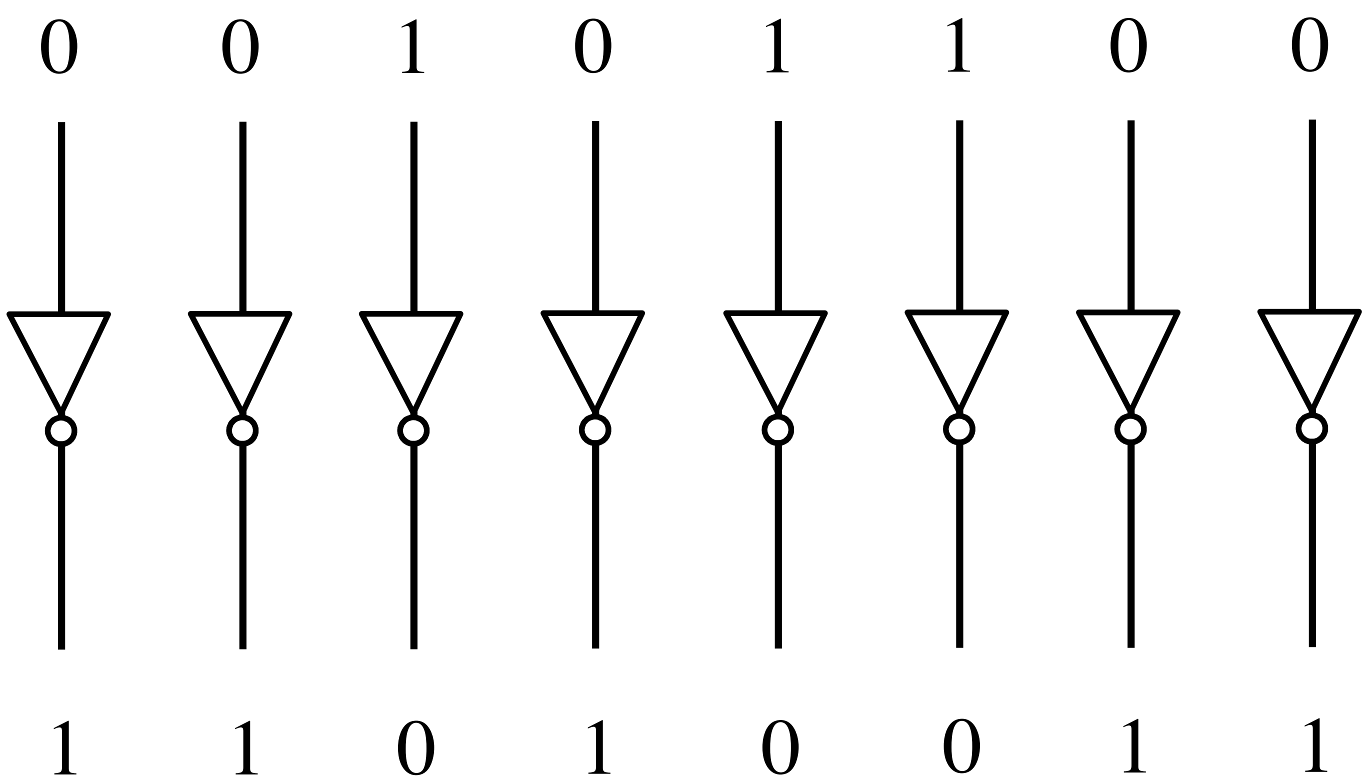
or

- 44 in 1's complement representation

Circuit for negating a number stored in 1's complement representation



Circuit for negating a number stored in 1's complement representation



This works in reverse too
(from negative to positive)

1's Complement Representation

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0
1	1	0	1	0	0	1	1

- 44

1's Complement Representation
(invert all the bits to negate the number)

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
1	1	0	1	0	0	1	1	- 44

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
0	0	1	0	1	1	0	0	+ 44

1's Complement Representation
(invert all the bits to negate the number)

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
1	1	0	1	0	0	1	1	- 44

211 (as unsigned)

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
0	0	1	0	1	1	0	0	+ 44

$44 = 255 - 211$ (as unsigned)

1's Complement Representation
(invert all the bits to negate the number)

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
1	1	0	1	0	0	1	1	- 44

- 44 in 1's complement representation

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
0	0	1	0	1	1	0	0	+ 44

+ 44 in 1's complement representation

Find the 1's complement of …

0 1 0 1

0 0 1 0

0 0 1 1

0 1 1 1

Find the 1's complement of …

0 1 0 1

1 0 1 0

0 0 1 0

1 1 0 1

0 0 1 1

1 1 0 0

0 1 1 1

1 0 0 0

Just flip 1's to 0's and vice versa.

A) Example of 1's complement addition

$$\begin{array}{r} (+5) \\ +\,(+2) \\ \hline (+7) \end{array} \qquad \begin{array}{r} 0\,1\,0\,1 \\ +\,0\,0\,1\,0 \\ \hline 0\,1\,1\,1 \end{array}$$

$b_3b_2b_1b_0$	1's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−7
1001	−6
1010	−5
1011	−4
1100	−3
1101	−2
1110	−1
1111	−0

[Figure 3.8 from the textbook]

A) Example of 1's complement addition

$$\begin{array}{r} (+5) \\ +\,(+2) \\ \hline (+7) \end{array} \qquad \begin{array}{r} 0\,1\,0\,1 \\ +\,0\,0\,1\,0 \\ \hline 0\,1\,1\,1 \end{array}$$

$b_3b_2b_1b_0$	1's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−7
1001	−6
1010	−5
1011	−4
1100	−3
1101	−2
1110	−1
1111	−0

B) Example of 1's complement addition

$$\begin{array}{r} (-5) \\ +\,(+2) \\ \hline (-3) \end{array} \qquad \begin{array}{r} 1\,0\,1\,0 \\ +\,0\,0\,1\,0 \\ \hline 1\,1\,0\,0 \end{array}$$

$b_3b_2b_1b_0$	1's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−7
1001	−6
1010	−5
1011	−4
1100	−3
1101	−2
1110	−1
1111	−0

[Figure 3.8 from the textbook]

B) Example of 1's complement addition

(- 5)	1 0 1 0
+ (+2)	+ 0 0 1 0
(- 3)	1 1 0 0

$b_3 b_2 b_1 b_0$	1's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−7
1001	−6
1010	−5
1011	−4
1100	−3
1101	−2
1110	−1
1111	−0

C) Example of 1's complement addition

$$\begin{array}{r} (+5) \\ +\,(-2) \\ \hline (+3) \end{array} \qquad \begin{array}{r} 0\,1\,0\,1 \\ +\,1\,1\,0\,1 \\ \hline 1\,0\,0\,1\,0 \end{array}$$

$b_3b_2b_1b_0$	1's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−7
1001	−6
1010	−5
1011	−4
1100	−3
1101	−2
1110	−1
1111	−0

[Figure 3.8 from the textbook]

C) Example of 1's complement addition

$$\begin{array}{r} (+5) \\ +(-2) \\ \hline (+3) \end{array} \qquad \begin{array}{r} 0\,1\,0\,1 \\ +\,1\,1\,0\,1 \\ \hline 1\,0\,0\,1\,0 \end{array}$$

$b_3b_2b_1b_0$	1's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−7
1001	−6
1010	−5
1011	−4
1100	−3
1101	−2
1110	−1
1111	−0

C) Example of 1's complement addition

$$\begin{array}{r} (+5) \\ +(-2) \\ \hline (+3) \end{array} \qquad \begin{array}{r} 0\,1\,0\,1 \\ +\,1\,1\,0\,1 \\ \hline 1\,0\,0\,1\,0 \end{array}$$

But this is 2!

$b_3b_2b_1b_0$	1's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−7
1001	−6
1010	−5
1011	−4
1100	−3
1101	−2
1110	−1
1111	−0

C) Example of 1's complement addition

$$\begin{array}{r} (+5) \\ +(-2) \\ \hline (+3) \end{array} \qquad \begin{array}{r} 0\,1\,0\,1 \\ +\,1\,1\,0\,1 \\ \hline 1\,0\,0\,1\,0 \\ 1 \\ \hline 0\,0\,1\,1 \end{array}$$

We need to perform one more addition to get the result.

$b_3b_2b_1b_0$	1's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−7
1001	−6
1010	−5
1011	−4
1100	−3
1101	−2
1110	−1
1111	−0

C) Example of 1's complement addition

$$
\begin{array}{rr}
(+5) & 0\,1\,0\,1 \\
+\,(-2) & +\;1\,1\,0\,1 \\
\hline
(+3) & 1\;\;0\,0\,1\,0 \\
 & 1 \\
\hline
 & 0\,0\,1\,1
\end{array}
$$

We need to perform one more addition to get the result.

$b_3b_2b_1b_0$	1's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−7
1001	−6
1010	−5
1011	−4
1100	−3
1101	−2
1110	−1
1111	−0

D) Example of 1's complement addition

$$\begin{array}{r} (-5) \\ +\,(-2) \\ \hline (-7) \end{array} \qquad \begin{array}{r} 1\,0\,1\,0 \\ +\,1\,1\,0\,1 \\ \hline 1\,0\,1\,1\,1 \end{array}$$

$b_3b_2b_1b_0$	1's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−7
1001	−6
1010	−5
1011	−4
1100	−3
1101	−2
1110	−1
1111	−0

[Figure 3.8 from the textbook]

D) Example of 1's complement addition

$$\begin{array}{rr} & (-5) \\ + & (-2) \\ \hline & (-7) \end{array} \qquad \begin{array}{rr} & 1\,0\,1\,0 \\ + & 1\,1\,0\,1 \\ \hline 1 & 0\,1\,1\,1 \end{array}$$

$b_3b_2b_1b_0$	1's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−7
1001	−6
1010	−5
1011	−4
1100	−3
1101	−2
1110	−1
1111	−0

D) Example of 1's complement addition

$$\begin{array}{rr} & (-5) \\ + & (-2) \\ \hline & (-7) \end{array} \qquad \begin{array}{rrrrr} & 1 & 0 & 1 & 0 \\ + & 1 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 1 & 1 \end{array}$$

But this is +7!

$b_3b_2b_1b_0$	1's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−7
1001	−6
1010	−5
1011	−4
1100	−3
1101	−2
1110	−1
1111	−0

D) Example of 1's complement addition

$$\begin{array}{rr} & (-5) \\ + & (-2) \\ \hline & (-7) \end{array} \qquad \begin{array}{rrrrrr} & & 1 & 0 & 1 & 0 \\ + & & 1 & 1 & 0 & 1 \\ \hline & 1 & 0 & 1 & 1 & 1 \\ & & & & & 1 \\ \hline & & 1 & 0 & 0 & 0 \end{array}$$

We need to perform one more addition to get the result.

$b_3b_2b_1b_0$	1's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−7
1001	−6
1010	−5
1011	−4
1100	−3
1101	−2
1110	−1
1111	−0

D) Example of 1's complement addition

(−5)	1 0 1 0
+ (−2)	+ 1 1 0 1
(−7)	1 0 1 1 1
	1
	1 0 0 0

We need to perform one more addition to get the result.

$b_3b_2b_1b_0$	1's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−7
1001	−6
1010	−5
1011	−4
1100	−3
1101	−2
1110	−1
1111	−0

2’s Complement

2’s complement
(subtract each digit from 1 and add 1 to the result)

Let K be the negative equivalent of an n-bit positive number P.

Then, in 2’s complement representation K is obtained by subtracting P from 2^n, namely

$$K = 2^n - P$$

2' s complement
(subtract each digit from 1 and add 1 to the result)

Let K be the negative equivalent of an 8-bit positive number P.

Then, in 2' s complement representation K is obtained by subtracting P from 2^8 , namely

$$K = 2^8 - P = 256 - P$$

2’s Complement Representation

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0
0	0	1	0	1	1	0	0

+ 44

2’s Complement Representation

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
0	0	1	0	1	1	0	0	+ 44

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
1	1	0	1	0	1	0	0	- 44

2's Complement Representation

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
0	0	1	0	1	1	0	0	+ 44
positive								

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
1	1	0	1	0	1	0	0	- 44
negative								

2's Complement Representation

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
0	0	1	0	1	1	0	0	+ 44

sign	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
1	1	0	1	0	1	0	0	- 44

212 = 256 - 44

Deriving 2’s complement

For a positive n-bit number P, let K_1 and K_2 denote its 1’s and 2’s complements, respectively.

$$K_1 = (2^n - 1) - P$$

$$K_2 = 2^n - P$$

Since $K_2 = K_1 + 1$, it is evident that in a logic circuit the 2’s complement can computed by inverting all bits of P and then adding 1 to the resulting 1’s-complement number.

Deriving 2's complement

For a positive 8-bit number P, let K_1 and K_2 denote its 1's and 2's complements, respectively.

$$K_1 = (2^n - 1) - P = 255 - P$$

$$K_2 = 2^n - P = 256 - P$$

Since $K_2 = K_1 + 1$, it is evident that in a logic circuit the 2's complement can computed by inverting all bits of P and then adding 1 to the resulting 1's-complement number.

Find the 2’s complement of …

0 1 0 1

0 0 1 0

0 1 0 0

0 1 1 1

Find the 2's complement of …

0 1 0 1	0 0 1 0
1 0 1 0	1 1 0 1
0 1 0 0	0 1 1 1
1 0 1 1	1 0 0 0

Invert all bits.

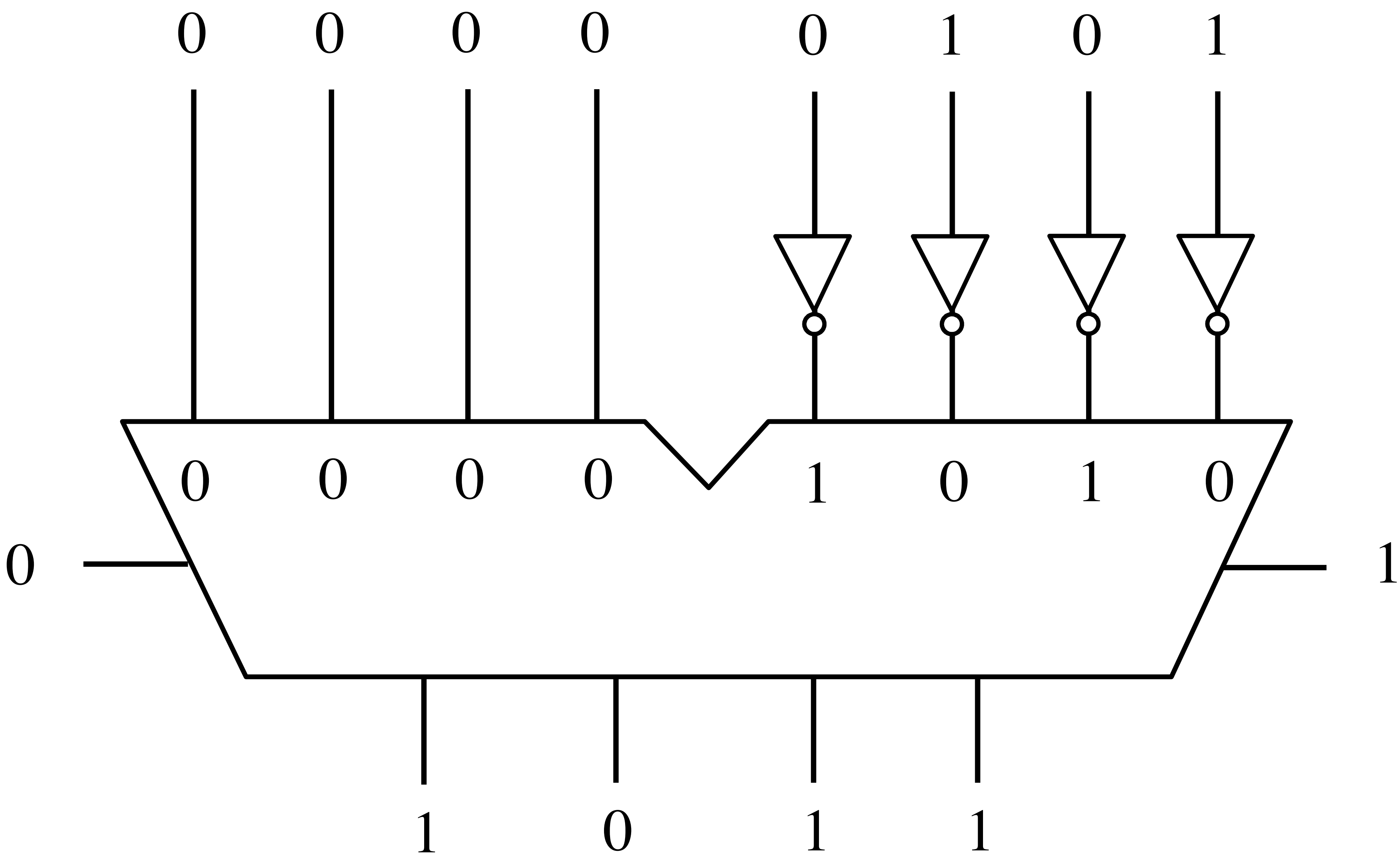
Find the 2's complement of …

$$\begin{array}{r} 0101 \\ 1010 \\ +\quad 1 \\ \hline 1011 \end{array} \qquad \begin{array}{r} 0010 \\ 1101 \\ +\quad 1 \\ \hline 1110 \end{array}$$

$$\begin{array}{r} 0100 \\ 1011 \\ +\quad 1 \\ \hline 1100 \end{array} \qquad \begin{array}{r} 0111 \\ 1000 \\ +\quad 1 \\ \hline 1001 \end{array}$$

Then add 1.

Circuit for inverting a number stored in 2's complement representation



Quick way (for a human) to find 2's complement

- Scan the binary number from right to left
- Copy all bits that are 0 from right to left
- Stop at the first 1
- Copy that 1 as well
- Invert all remaining bits

Find the 2’s complement of …

0 1 0 1

0 0 1 0

0 1 0 0

0 1 1 1

Find the 2's complement of …

0 1 0 1

. . . .

0 0 1 0

. . . 0

0 1 0 0

. . 0 0

0 1 1 1

. . . .

Copy all bits that are 0 from right to left.

Find the 2's complement of …

0 1 0 1
. . . 1

0 0 1 0
. . 1 0

0 1 0 0
. 1 0 0

0 1 1 1
. . . 1

Stop at the first 1. Copy that 1 as well.

Find the 2’s complement of …

0 1 0 1
1 0 1 1

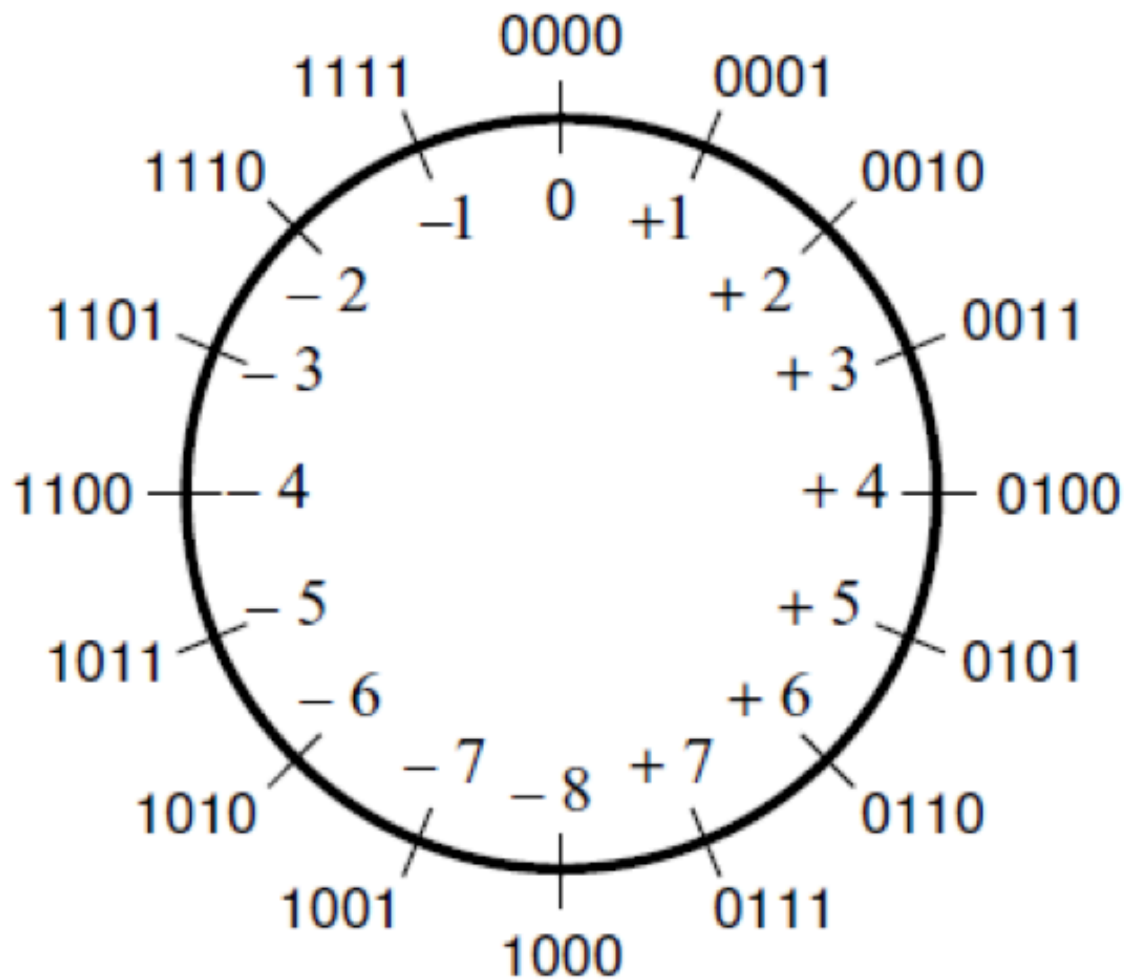
0 0 1 0
1 1 1 0

0 1 0 0
1 1 0 0

0 1 1 1
1 0 0 1

Invert all remaining bits.

The number circle for 2's complement



[Figure 3.11a from the textbook]

A) Example of 2's complement addition

$$\begin{array}{rr} & (+5) \\ + & (+2) \\ \hline & (+7) \end{array} \qquad \begin{array}{rr} & 0\,1\,0\,1 \\ + & 0\,0\,1\,0 \\ \hline & 0\,1\,1\,1 \end{array}$$

$b_3b_2b_1b_0$	2's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−8
1001	−7
1010	−6
1011	−5
1100	−4
1101	−3
1110	−2
1111	−1

[Figure 3.9 from the textbook]

B) Example of 2's complement addition

$$\begin{array}{rr} (-5) & 1\,0\,1\,1 \\ +\;(+2) & +\;0\,0\,1\,0 \\ \hline (-3) & 1\,1\,0\,1 \end{array}$$

$b_3b_2b_1b_0$	2's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−8
1001	−7
1010	−6
1011	−5
1100	−4
1101	−3
1110	−2
1111	−1

[Figure 3.9 from the textbook]

C) Example of 2's complement addition

$$
\begin{array}{rr}
 & (+5) \\
+ & (-2) \\
\hline
 & (+3)
\end{array}
\qquad\qquad
\begin{array}{rr}
 & 0\,1\,0\,1 \\
+ & 1\,1\,1\,0 \\
\hline
1 & 0\,0\,1\,1
\end{array}
$$

ignore

$b_3b_2b_1b_0$	2's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−8
1001	−7
1010	−6
1011	−5
1100	−4
1101	−3
1110	−2
1111	−1

[Figure 3.9 from the textbook]

D) Example of 2’s complement addition

(–5)	1 0 1 1
+ (–2)	+ 1 1 1 0
(–7)	1 1 0 0 1

ignore

$b_3b_2b_1b_0$	2’s complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−8
1001	−7
1010	−6
1011	−5
1100	−4
1101	−3
1110	−2
1111	−1

[Figure 3.9 from the textbook]

Naming Ambiguity: 2's Complement

2's complement has two different meanings:

- **representation for signed integer numbers**

- **algorithm for computing the 2's complement (regardless of the representation of the number)**

Naming Ambiguity: 2's Complement

2's complement has two different meanings:

- representation for signed integer numbers
  in 2's complement

- algorithm for computing the 2's complement (regardless of the representation of the number)
  take the 2's complement

Example of 2’s complement <u>subtraction</u>

$$
\begin{array}{r} (+5) \\ -\ (+2) \\ \hline (+3) \end{array}
\qquad
\begin{array}{r} 0\,1\,0\,1 \\ -\ 0\,0\,1\,0 \\ \hline \end{array}
\quad \Longrightarrow \quad
\begin{array}{r} 0\,1\,0\,1 \\ +\ 1\,1\,1\,0 \\ \hline 1\,0\,0\,1\,1 \end{array}
$$

ignore

$\Longrightarrow$ means take the 2's complement

[Figure 3.10 from the textbook]

Example of 2’s complement <u>subtraction</u>

$$
\begin{array}{r}
(+5) \\
-\ (+2) \\
\hline
(+3)
\end{array}
\qquad
\begin{array}{r}
0\,1\,0\,1 \\
-\ 0\,0\,1\,0 \\
\hline
\end{array}
\qquad \Longrightarrow \qquad
\begin{array}{r}
0\,1\,0\,1 \\
+\ 1\,1\,1\,0 \\
\hline
1\,0\,0\,1\,1
\end{array}
$$

ignore

Notice that the minus changes to a plus.

$\Longrightarrow$ means take the 2's complement

[Figure 3.10 from the textbook]

Example of 2' s complement <u>subtraction</u>

$$\begin{array}{r} (+5) \\ -\ (+2) \\ \hline (+3) \end{array} \qquad \begin{array}{r} 0\,1\,0\,1 \\ -\ 0\,0\,1\,0 \\ \hline \end{array} \quad \Longrightarrow \quad \begin{array}{r} 0\,1\,0\,1 \\ +\ 1\,1\,1\,0 \\ \hline 1\,0\,0\,1\,1 \end{array}$$

ignore

$b_3b_2b_1b_0$	2's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−8
1001	−7
1010	−6
1011	−5
1100	−4
1101	−3
1110	−2
1111	−1

[Figure 3.10 from the textbook]

Example of 2's complement <u>subtraction</u>

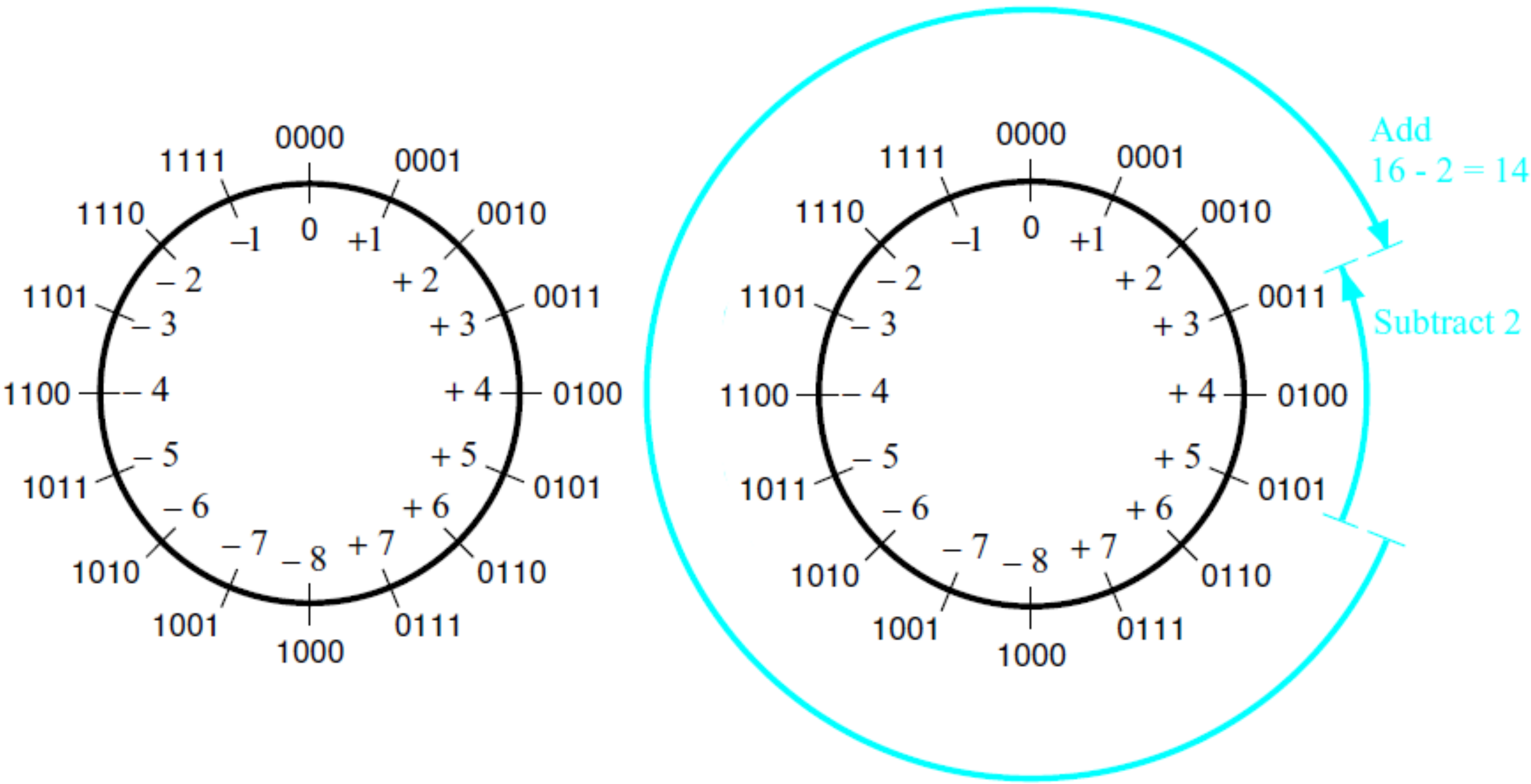
$$
\begin{array}{r} (+5) \\ -\ (+2) \\ \hline (+3) \end{array}
\qquad
\begin{array}{r} 0\,1\,0\,1 \\ -\ 0\,0\,1\,0 \\ \hline \end{array}
\qquad \Longrightarrow \qquad
\begin{array}{r} 0\,1\,0\,1 \\ +\ 1\,1\,1\,0 \\ \hline 1\ 0\,0\,1\,1 \end{array}
$$

ignore

$b_3b_2b_1b_0$	2's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−8
1001	−7
1010	−6
1011	−5
1100	−4
1101	−3
1110	−2
1111	−1

[Figure 3.10 from the textbook]

Graphical interpretation of four-bit 2's complement numbers



(a) The number circle (b) Subtracting 2 by adding its 2's complement

[Figure 3.11 from the textbook]

Example of 2’s complement subtraction

$$\begin{array}{r} (-5) \\ -\ (+2) \\ \hline (-7) \end{array} \qquad \begin{array}{r} 1\,0\,1\,1 \\ -\ 0\,0\,1\,0 \\ \hline \end{array} \qquad \Longrightarrow \qquad \begin{array}{r} 1\,0\,1\,1 \\ +\ 1\,1\,1\,0 \\ \hline 1\ 1\,0\,0\,1 \end{array}$$

ignore

$b_3b_2b_1b_0$	2’s complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−8
1001	−7
1010	−6
1011	−5
1100	−4
1101	−3
1110	−2
1111	−1

[Figure 3.10 from the textbook]

Example of 2's complement subtraction

$$\begin{array}{r} (+5) \\ -\ (-2) \\ \hline (+7) \end{array} \qquad \begin{array}{r} 0\,1\,0\,1 \\ -\ 1\,1\,1\,0 \\ \hline \end{array} \qquad \Longrightarrow \qquad \begin{array}{r} 0\,1\,0\,1 \\ +\ 0\,0\,1\,0 \\ \hline 0\,1\,1\,1 \end{array}$$

$b_3b_2b_1b_0$	2's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−8
1001	−7
1010	−6
1011	−5
1100	−4
1101	−3
1110	−2
1111	−1

[Figure 3.10 from the textbook]

Example of 2’s complement subtraction

$$\begin{array}{r} (-5) \\ -\ (-2) \\ \hline (-3) \end{array} \qquad \begin{array}{r} 1\,0\,1\,1 \\ -\ 1\,1\,1\,0 \\ \hline \end{array} \quad \Longrightarrow \quad \begin{array}{r} 1\,0\,1\,1 \\ +\ 0\,0\,1\,0 \\ \hline 1\,1\,0\,1 \end{array}$$

$b_3b_2b_1b_0$	2’s complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−8
1001	−7
1010	−6
1011	−5
1100	−4
1101	−3
1110	−2
1111	−1

[Figure 3.10 from the textbook]

Taking the 2' s complement negates the number

decimal	$b_3\, b_2\, b_1\, b_0$	take the 2's complement	$b_3\, b_2\, b_1\, b_0$	decimal
+7	0111	$\Longrightarrow$	1001	−7
+6	0110	$\Longrightarrow$	1010	−6
+5	0101	$\Longrightarrow$	1011	−5
+4	0100	$\Longrightarrow$	1100	−4
+3	0011	$\Longrightarrow$	1101	−3
+2	0010	$\Longrightarrow$	1110	−2
+1	0001	$\Longrightarrow$	1111	−1
+0	0000	$\Longrightarrow$	0000	+0
−8	1000	$\Longrightarrow$	1000	−8
−7	1001	$\Longrightarrow$	0111	+7
−6	1010	$\Longrightarrow$	0110	+6
−5	1011	$\Longrightarrow$	0101	+5
−4	1100	$\Longrightarrow$	0100	+4
−3	1101	$\Longrightarrow$	0011	+3
−2	1110	$\Longrightarrow$	0010	+2
−1	1111	$\Longrightarrow$	0001	+1

Taking the 2’s complement negates the number

decimal	$b_3\,b_2\,b_1\,b_0$	take the 2's complement	$b_3\,b_2\,b_1\,b_0$	decimal
+7	0111	$\Longrightarrow$	1001	−7
+6	0110	$\Longrightarrow$	1010	−6
+5	0101	$\Longrightarrow$	1011	−5
+4	0100	$\Longrightarrow$	1100	−4
+3	0011	$\Longrightarrow$	1101	−3
+2	0010	$\Longrightarrow$	1110	−2
+1	0001	$\Longrightarrow$	1111	−1
+0	0000	$\Longrightarrow$	0000	+0
−8	1000	$\Longrightarrow$	1000	−8
−7	1001	$\Longrightarrow$	0111	+7
−6	1010	$\Longrightarrow$	0110	+6
−5	1011	$\Longrightarrow$	0101	+5
−4	1100	$\Longrightarrow$	0100	+4
−3	1101	$\Longrightarrow$	0011	+3
−2	1110	$\Longrightarrow$	0010	+2
−1	1111	$\Longrightarrow$	0001	+1

This is the only exception

Taking the 2's complement negates the number

decimal	$b_3\, b_2\, b_1\, b_0$	take the 2's complement	$b_3\, b_2\, b_1\, b_0$	decimal
+7	0111	$\Longrightarrow$	1001	−7
+6	0110	$\Longrightarrow$	1010	−6
+5	0101	$\Longrightarrow$	1011	−5
+4	0100	$\Longrightarrow$	1100	−4
+3	0011	$\Longrightarrow$	1101	−3
+2	0010	$\Longrightarrow$	1110	−2
+1	0001	$\Longrightarrow$	1111	−1
+0	0000	$\Longrightarrow$	0000	+0
−8	1000	$\Longrightarrow$	1000	−8
−7	1001	$\Longrightarrow$	0111	+7
−6	1010	$\Longrightarrow$	0110	+6
−5	1011	$\Longrightarrow$	0101	+5
−4	1100	$\Longrightarrow$	0100	+4
−3	1101	$\Longrightarrow$	0011	+3
−2	1110	$\Longrightarrow$	0010	+2
−1	1111	$\Longrightarrow$	0001	+1

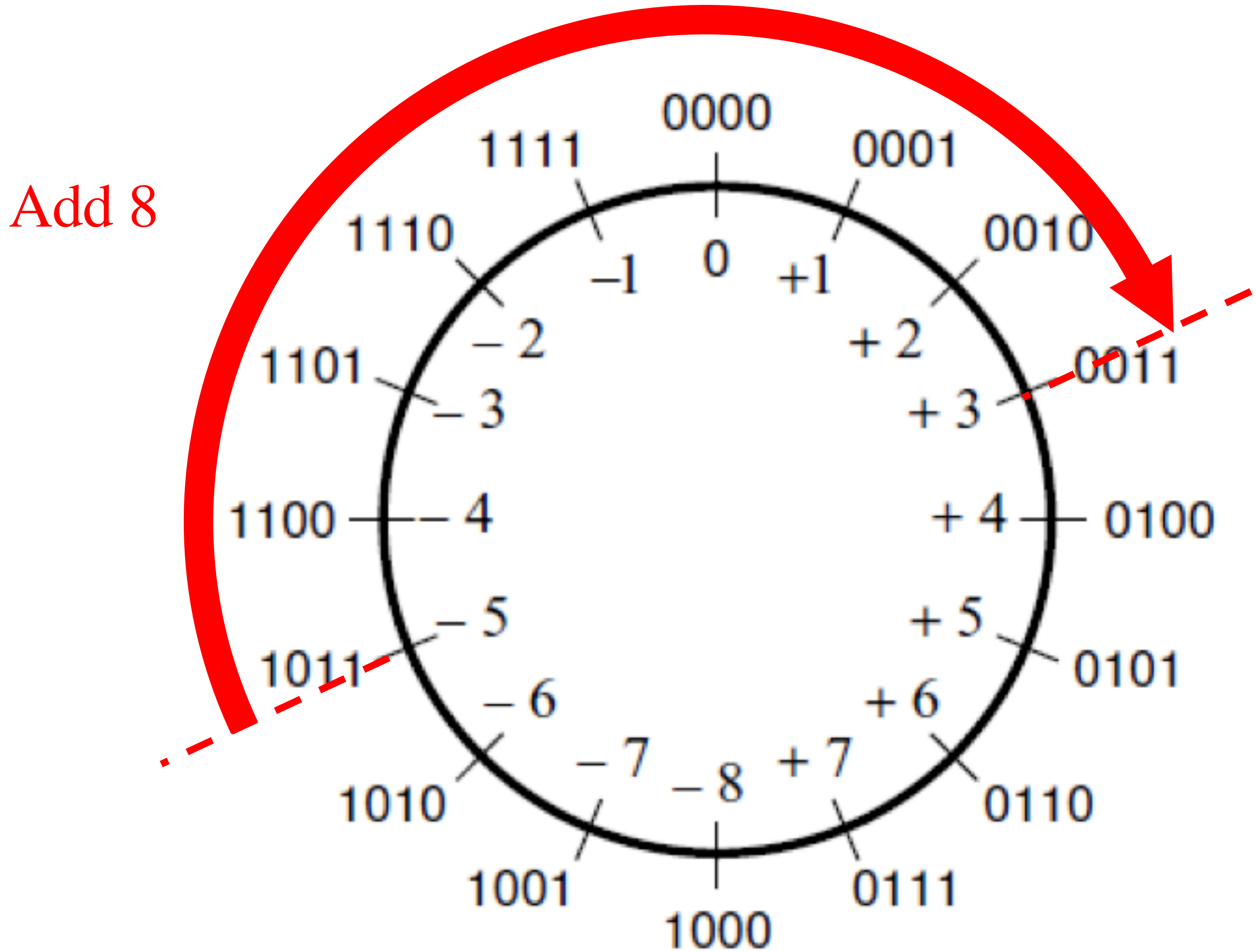
And this one too.

But that exception does not matter

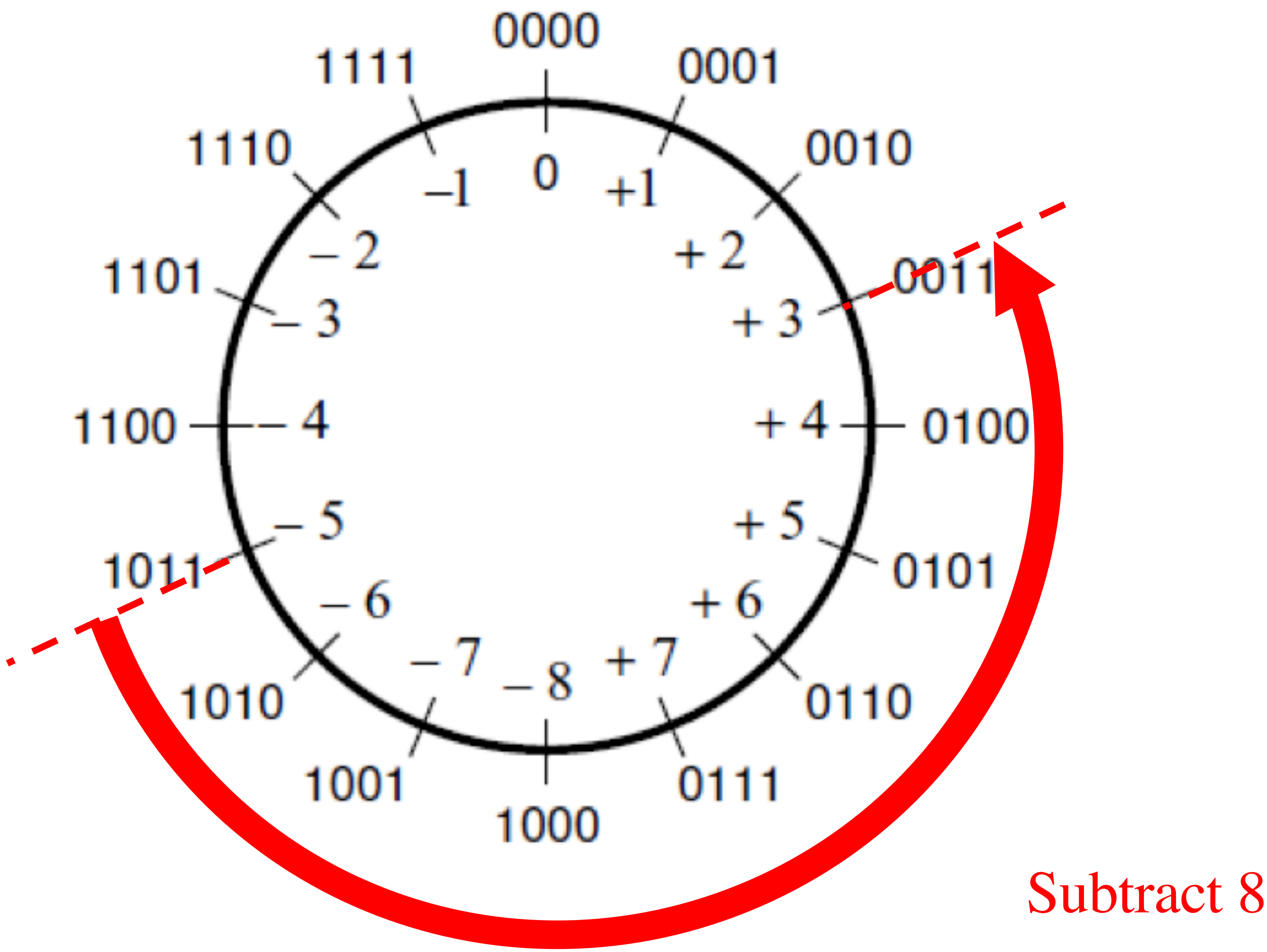
$$\begin{array}{r} (-5) \\ -\ (-8) \\ \hline (+3) \end{array} \qquad \begin{array}{r} 1\,0\,1\,1 \\ -\ 1\,0\,0\,0 \\ \hline \end{array} \qquad \Longrightarrow \qquad \begin{array}{r} 1\,0\,1\,1 \\ +\ 1\,0\,0\,0 \\ \hline 1\,0\,0\,1\,1 \end{array}$$

ignore

But that exception does not matter



But that exception does not matter

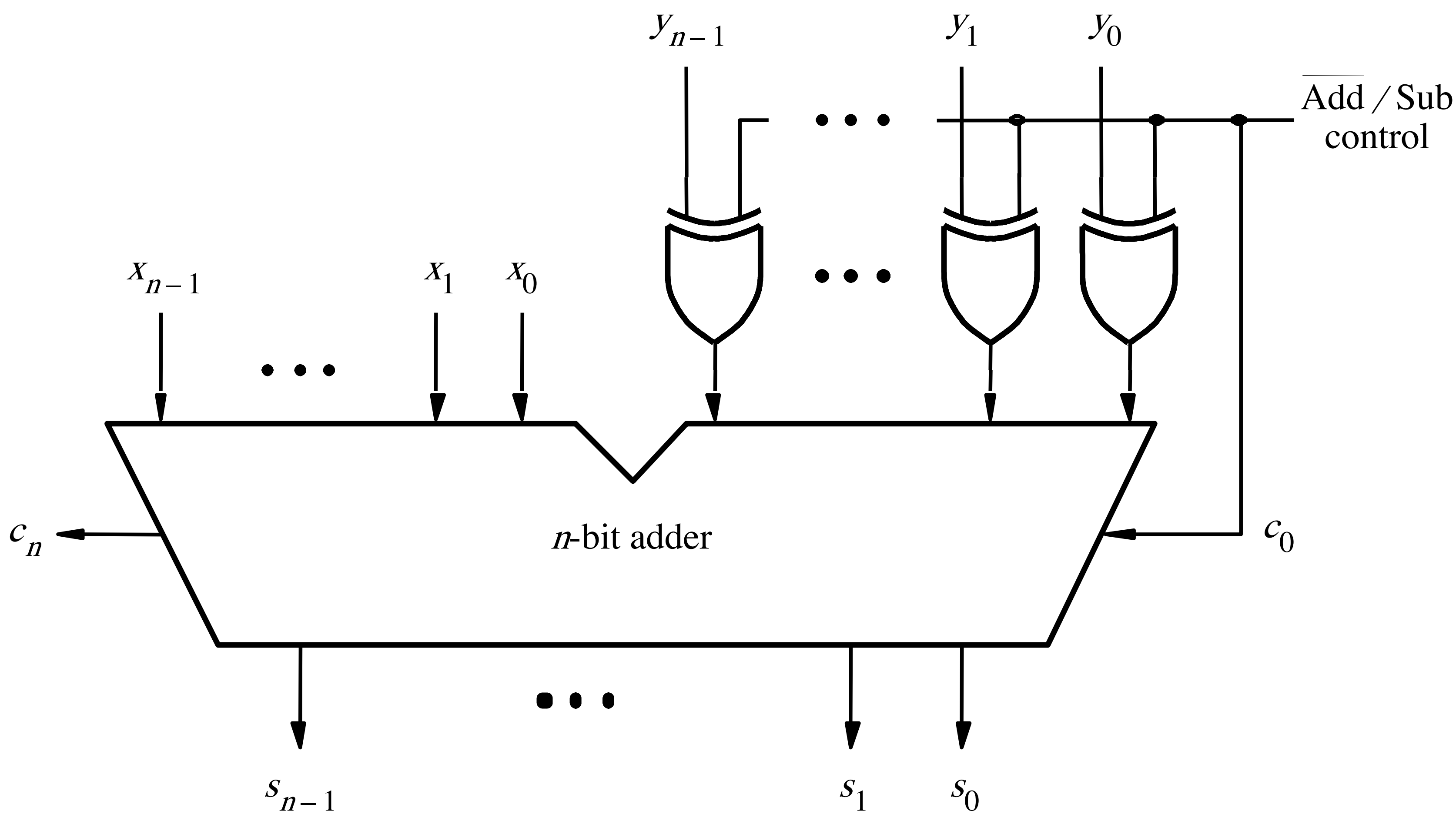


Take-Home Message

- Subtraction can be performed by simply adding the 2's complement of the second number, regardless of the signs of the two numbers.

- Thus, the same adder circuit can be used to perform <u>both addition and subtraction</u> !!!

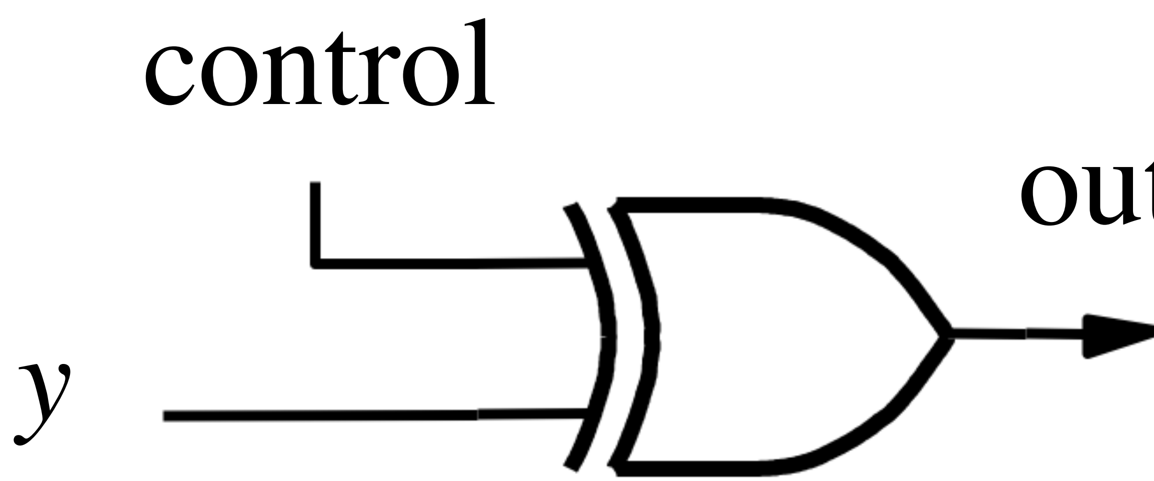
Adder/subtractor unit



[Figure 3.12 from the textbook]

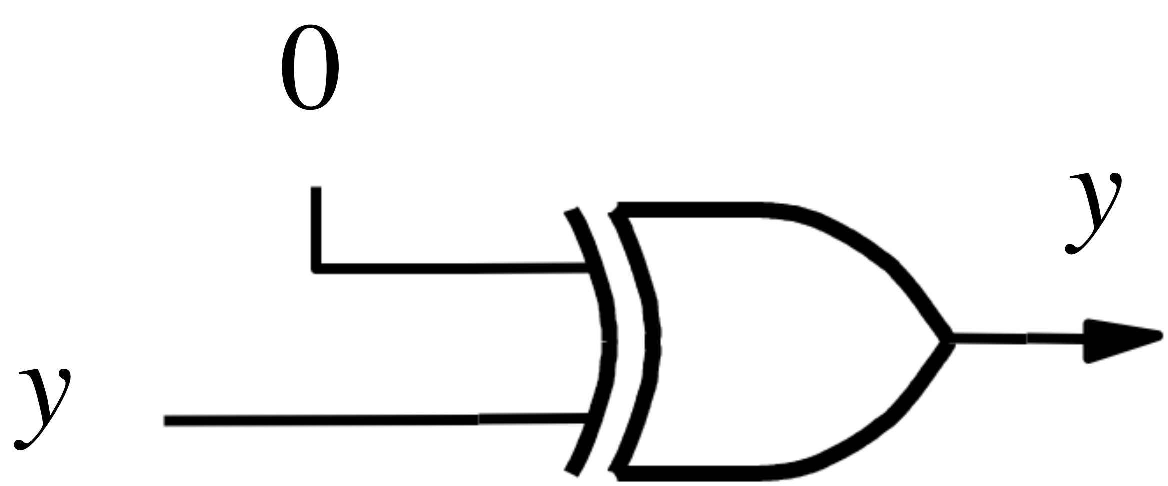
XOR Tricks

control	y	out
0	0	0
0	1	1
1	0	1
1	1	0



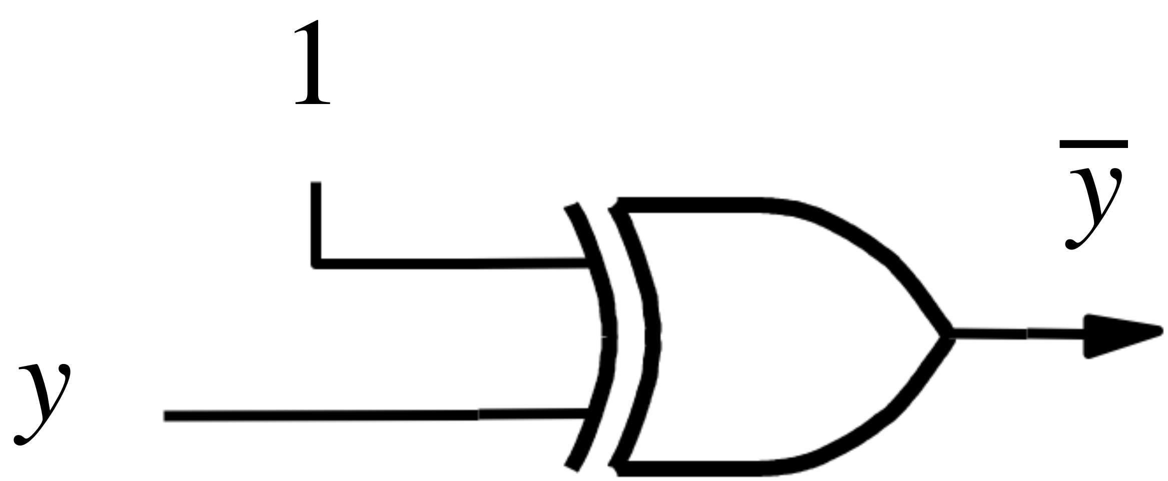
XOR as a repeater

control	y	out
0	0	0
0	1	1

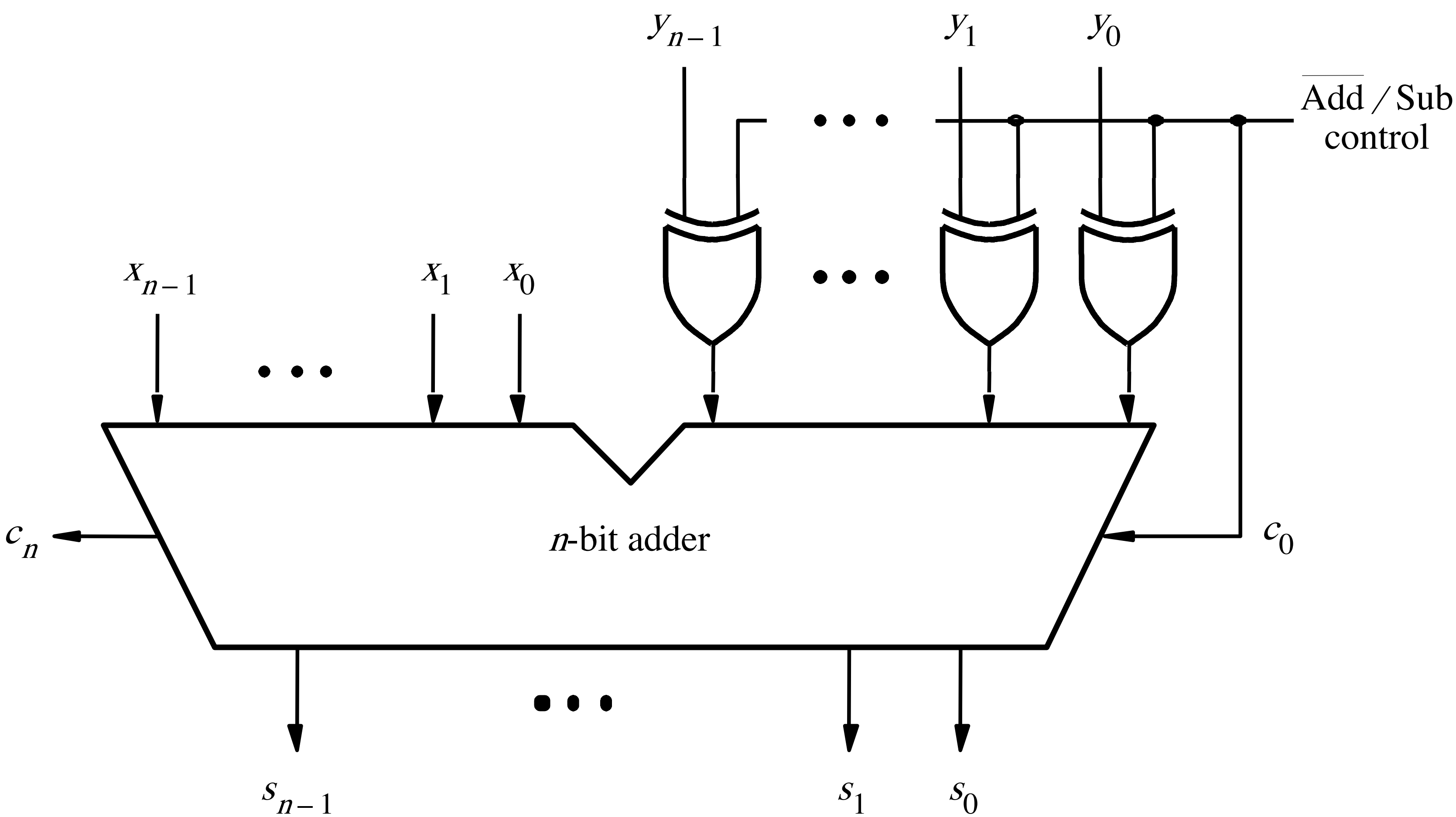


XOR as an inverter

control	y	out
1	0	1
1	1	0

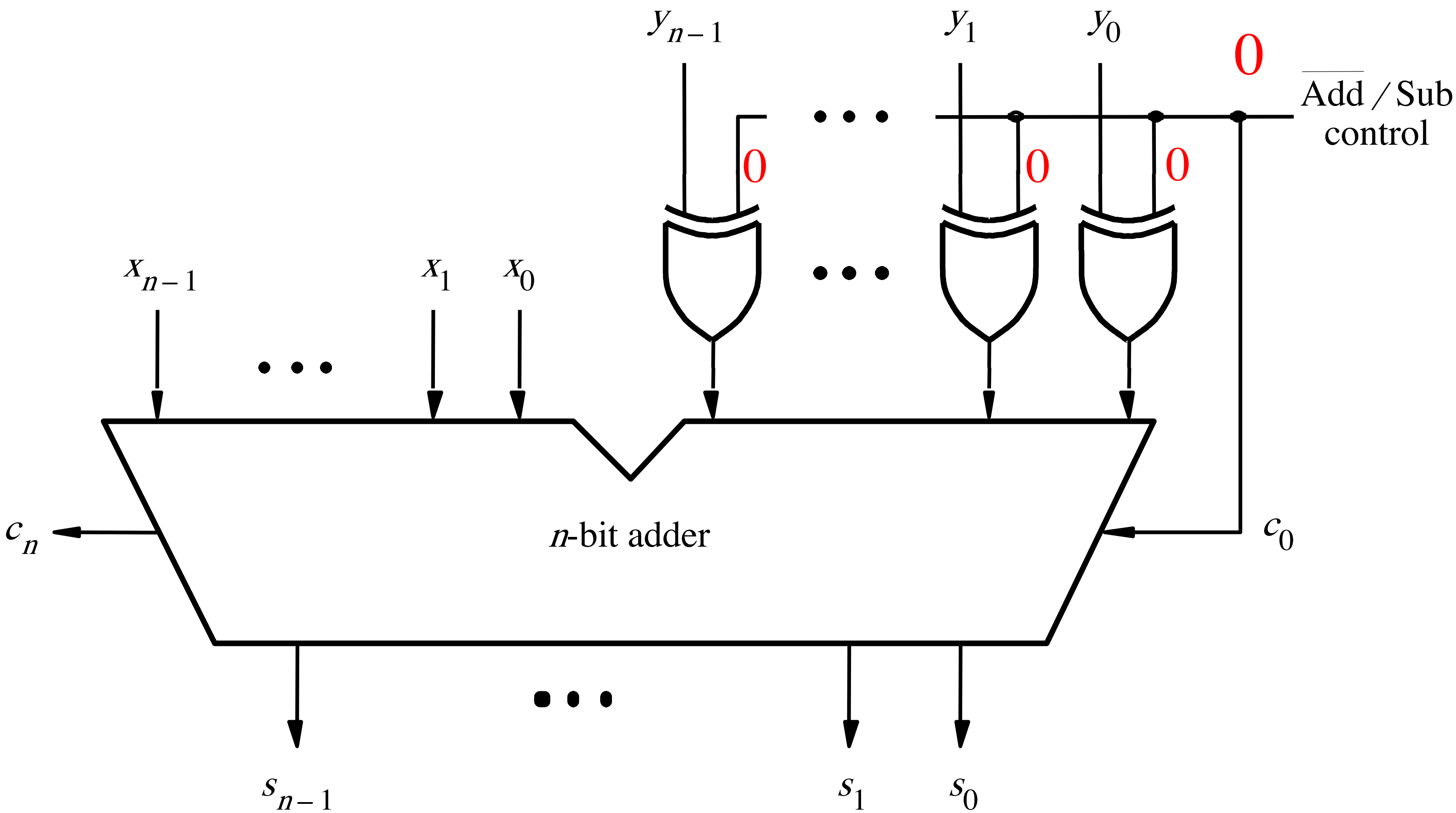


Addition: when control = 0



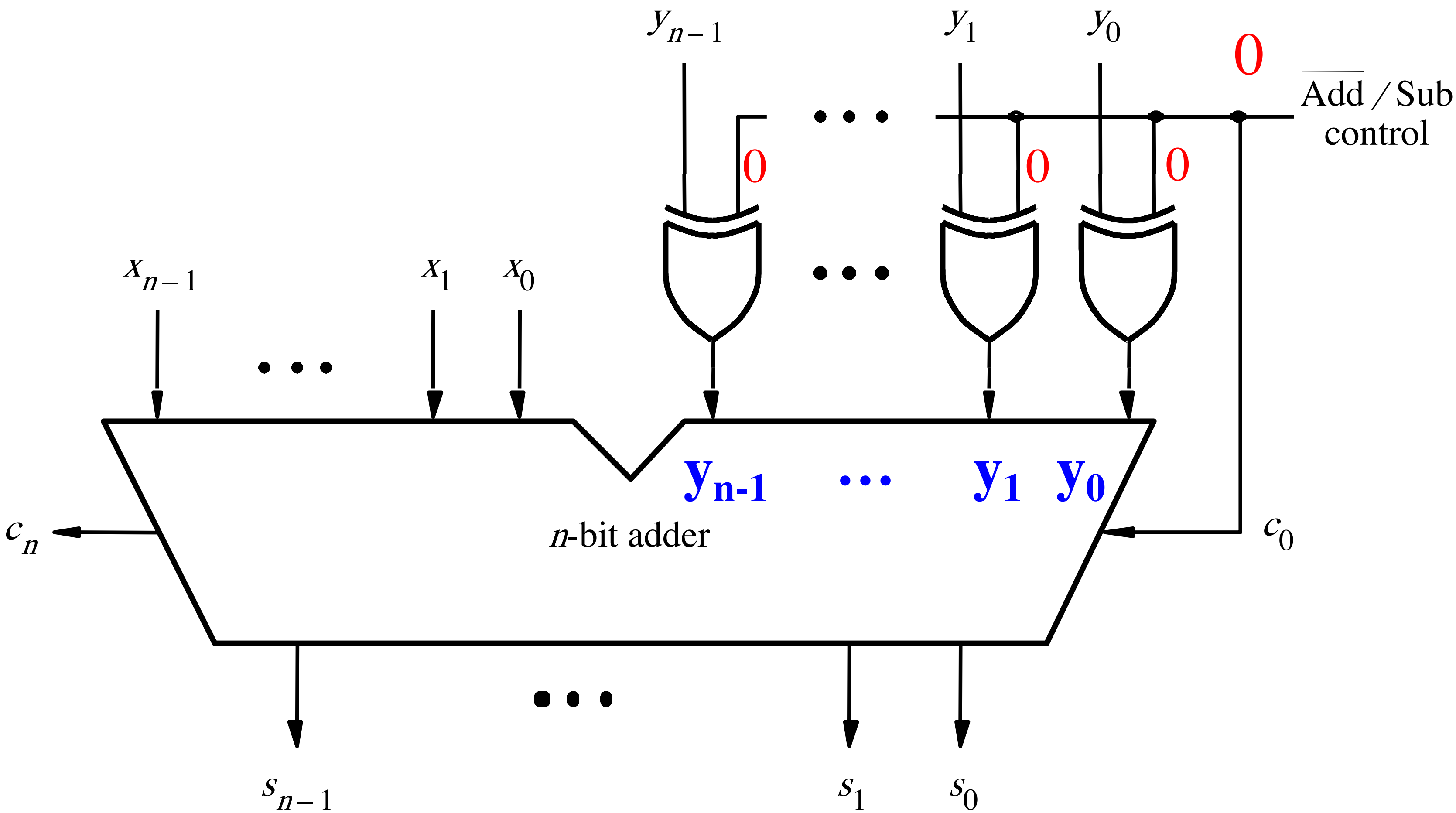
[Figure 3.12 from the textbook]

Addition: when control = 0



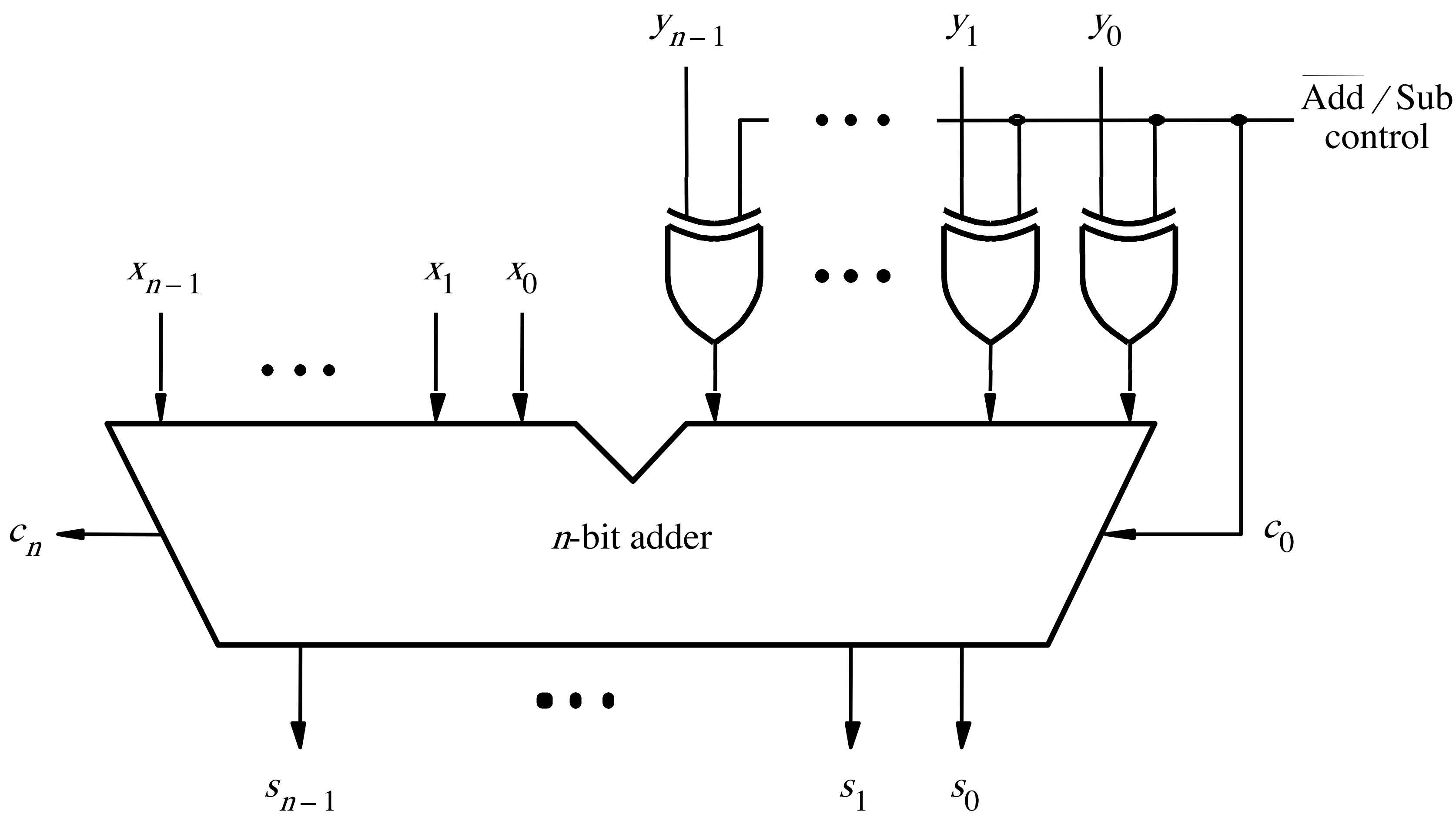
[Figure 3.12 from the textbook]

Addition: when control = 0



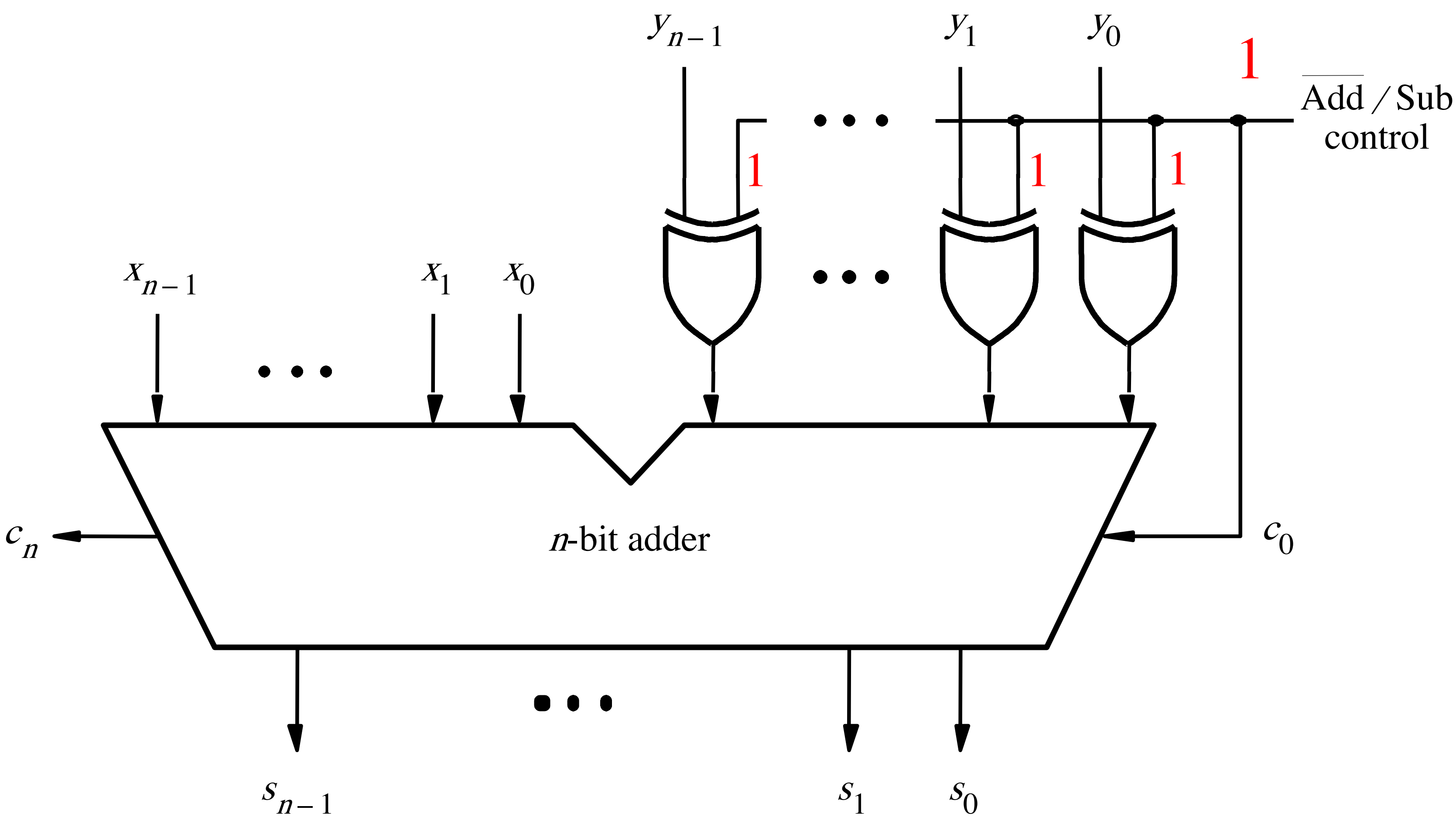
[Figure 3.12 from the textbook]

Subtraction: when control = 1



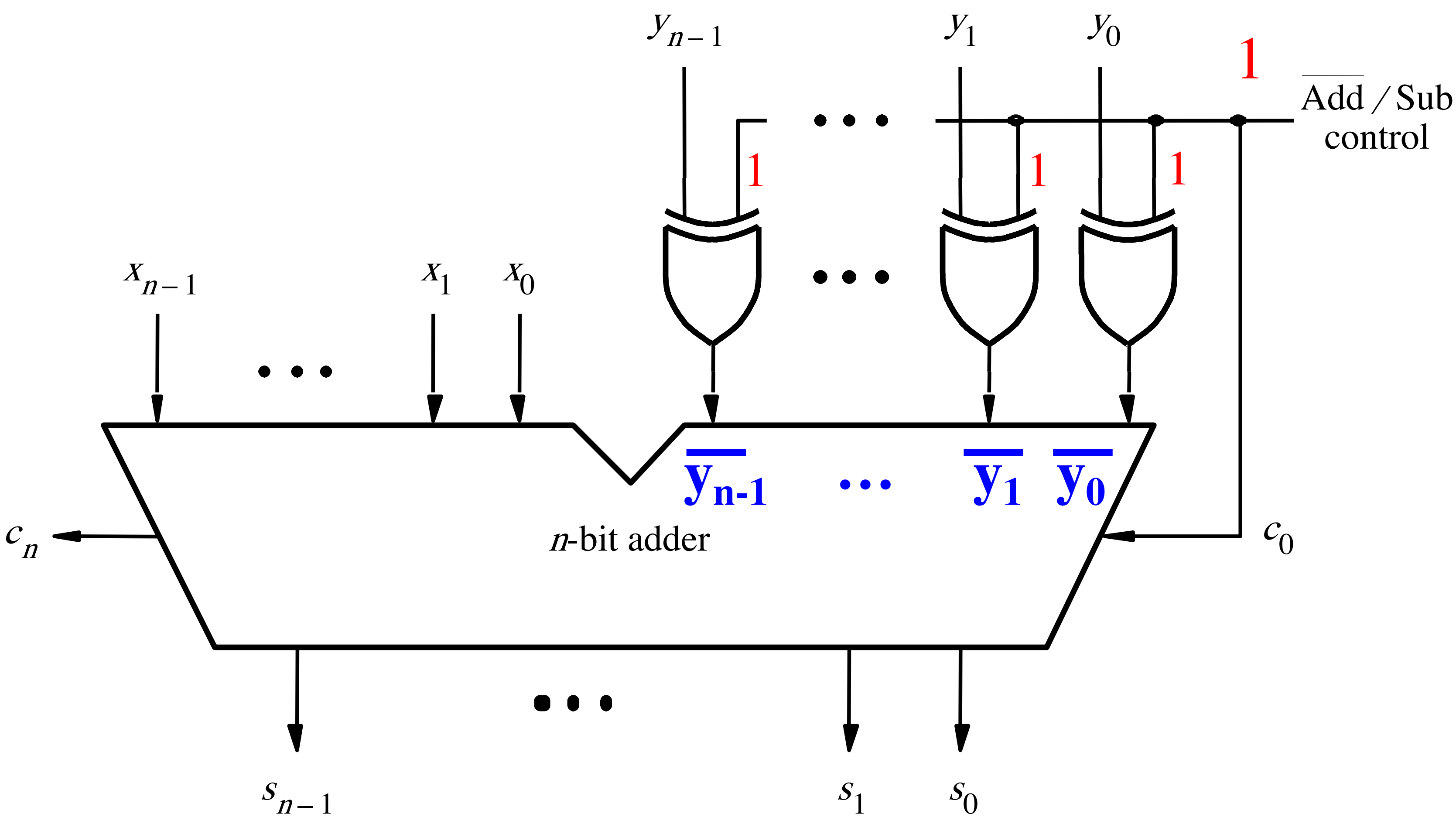
[Figure 3.12 from the textbook]

Subtraction: when control = 1



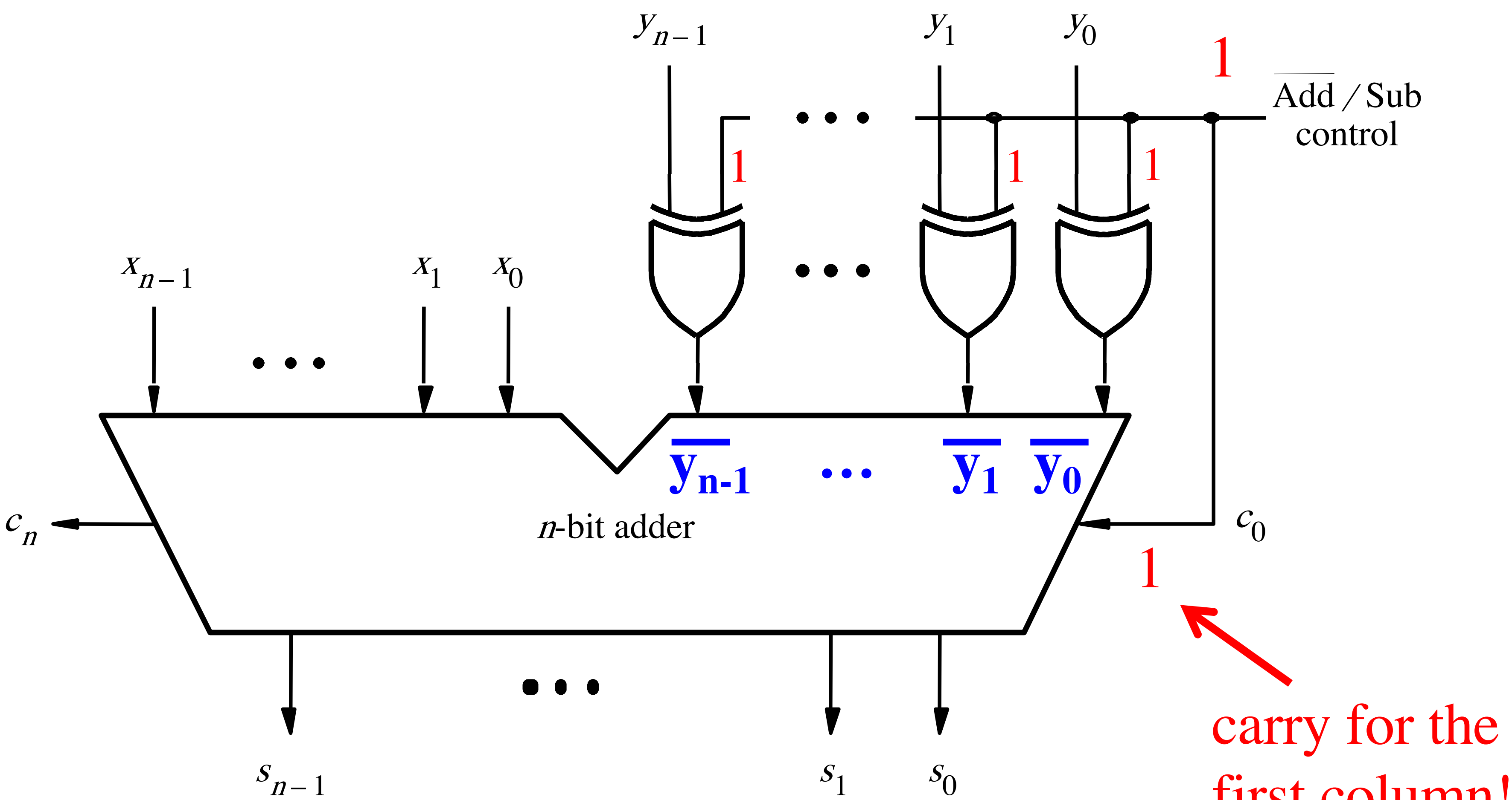
[Figure 3.12 from the textbook]

Subtraction: when control = 1



[Figure 3.12 from the textbook]

Subtraction: when control = 1



[Figure 3.12 from the textbook]

Examples of determination of overflow

$$\begin{array}{r} (+7) \\ +\ (+2) \\ \hline (+9) \end{array} \qquad \begin{array}{r} 0111 \\ +\ 0010 \\ \hline 1001 \end{array} \qquad\qquad \begin{array}{r} (-7) \\ +\ (+2) \\ \hline (-5) \end{array} \qquad \begin{array}{r} 1001 \\ +\ 0010 \\ \hline 1011 \end{array}$$

$$\begin{array}{r} (+7) \\ +\ (-2) \\ \hline (+5) \end{array} \qquad \begin{array}{r} 0111 \\ +\ 1110 \\ \hline 10101 \end{array} \qquad\qquad \begin{array}{r} (-7) \\ +\ (-2) \\ \hline (-9) \end{array} \qquad \begin{array}{r} 1001 \\ +\ 1110 \\ \hline 10111 \end{array}$$

[Figure 3.13 from the textbook]

Examples of determination of overflow

$$\begin{array}{r} (+7) \\ +\ (+2) \\ \hline (+9) \end{array} \qquad \begin{array}{rr} & 01100 \\ & 0111 \\ + & 0010 \\ \hline & 1001 \end{array}$$

$$\begin{array}{r} (-7) \\ +\ (+2) \\ \hline (-5) \end{array} \qquad \begin{array}{rr} & 00000 \\ & 1001 \\ + & 0010 \\ \hline & 1011 \end{array}$$

$$\begin{array}{r} (+7) \\ +\ (-2) \\ \hline (+5) \end{array} \qquad \begin{array}{rr} & 11100 \\ & 0111 \\ + & 1110 \\ \hline & 10101 \end{array}$$

$$\begin{array}{r} (-7) \\ +\ (-2) \\ \hline (-9) \end{array} \qquad \begin{array}{rr} & 10000 \\ & 1001 \\ + & 1110 \\ \hline & 10111 \end{array}$$

Include the carry bits: $c_4\ c_3\ c_2\ c_1\ c_0$

Examples of determination of overflow

$$
\begin{array}{rr}
 & \boxed{0\ 1}\ 1\ 0\ 0 \\
(+7) & 0\ 1\ 1\ 1 \\
+\ (+2) & +\quad 0\ 0\ 1\ 0 \\
\hline
(+9) & 1\ 0\ 0\ 1
\end{array}
\qquad\qquad
\begin{array}{rr}
 & \boxed{0\ 0}\ 0\ 0\ 0 \\
(-7) & 1\ 0\ 0\ 1 \\
+\ (+2) & +\quad 0\ 0\ 1\ 0 \\
\hline
(-5) & 1\ 0\ 1\ 1
\end{array}
$$

$$
\begin{array}{rr}
 & \boxed{1\ 1}\ 1\ 0\ 0 \\
(+7) & 0\ 1\ 1\ 1 \\
+\ (-2) & +\quad 1\ 1\ 1\ 0 \\
\hline
(+5) & 1\ 0\ 1\ 0\ 1
\end{array}
\qquad\qquad
\begin{array}{rr}
 & \boxed{1\ 0}\ 0\ 0\ 0 \\
(-7) & 1\ 0\ 0\ 1 \\
+\ (-2) & +\quad 1\ 1\ 1\ 0 \\
\hline
(-9) & 1\ 0\ 1\ 1\ 1
\end{array}
$$

Include the carry bits: $\boxed{c_4\ c_3}\ c_2\ c_1\ c_0$

Examples of determination of overflow

$c_4 = 0$
$c_3 = 1$

$$\begin{array}{r} (+7) \\ +\ (+2) \\ \hline (+9) \end{array} \qquad \begin{array}{r} \boxed{0\ 1}\,1\ 0\ 0 \\ 0\ 1\ 1\ 1 \\ +\quad 0\ 0\ 1\ 0 \\ \hline 1\ 0\ 0\ 1 \end{array}$$

$c_4 = 0$
$c_3 = 0$

$$\begin{array}{r} (-7) \\ +\ (+2) \\ \hline (-5) \end{array} \qquad \begin{array}{r} \boxed{0\ 0}\,0\ 0\ 0 \\ 1\ 0\ 0\ 1 \\ +\quad 0\ 0\ 1\ 0 \\ \hline 1\ 0\ 1\ 1 \end{array}$$

$c_4 = 1$
$c_3 = 1$

$$\begin{array}{r} (+7) \\ +\ (-2) \\ \hline (+5) \end{array} \qquad \begin{array}{r} \boxed{1\ 1}\,1\ 0\ 0 \\ 0\ 1\ 1\ 1 \\ +\quad 1\ 1\ 1\ 0 \\ \hline 1\ 0\ 1\ 0\ 1 \end{array}$$

$c_4 = 1$
$c_3 = 0$

$$\begin{array}{r} (-7) \\ +\ (-2) \\ \hline (-9) \end{array} \qquad \begin{array}{r} \boxed{1\ 0}\,0\ 0\ 0 \\ 1\ 0\ 0\ 1 \\ +\quad 1\ 1\ 1\ 0 \\ \hline 1\ 0\ 1\ 1\ 1 \end{array}$$

Include the carry bits: $\boxed{c_4\ c_3}\, c_2\ c_1\ c_0$

Examples of determination of overflow

$c_4 = 0$
$c_3 = 1$

$$\begin{array}{r} (+7) \\ +\ (+2) \\ \hline (+9) \end{array} \qquad \begin{array}{r} \boxed{0\,1}\,1\,0\,0 \\ 0\,1\,1\,1 \\ +\quad 0\,0\,1\,0 \\ \hline 1\,0\,0\,1 \end{array}$$

$$\begin{array}{r} (-7) \\ +\ (+2) \\ \hline (-5) \end{array} \qquad \begin{array}{r} \boxed{0\,0}\,0\,0\,0 \\ 1\,0\,0\,1 \\ +\quad 0\,0\,1\,0 \\ \hline 1\,0\,1\,1 \end{array}$$

$c_4 = 0$
$c_3 = 0$

$c_4 = 1$
$c_3 = 1$

$$\begin{array}{r} (+7) \\ +\ (-2) \\ \hline (+5) \end{array} \qquad \begin{array}{r} \boxed{1\,1}\,1\,0\,0 \\ 0\,1\,1\,1 \\ +\quad 1\,1\,1\,0 \\ \hline 1\,0\,1\,0\,1 \end{array}$$

$$\begin{array}{r} (-7) \\ +\ (-2) \\ \hline (-9) \end{array} \qquad \begin{array}{r} \boxed{1\,0}\,0\,0\,0 \\ 1\,0\,0\,1 \\ +\quad 1\,1\,1\,0 \\ \hline 1\,0\,1\,1\,1 \end{array}$$

$c_4 = 1$
$c_3 = 0$

Overflow occurs only in these two cases.

Examples of determination of overflow

$c_4 = 0$
$c_3 = 1$

$$\begin{array}{r} (+7) \\ +\ (+2) \\ \hline (+9) \end{array} \qquad \begin{array}{rr} & \boxed{0\ 1}\,1\ 0\ 0 \\ & 0\ 1\ 1\ 1 \\ + & 0\ 0\ 1\ 0 \\ \hline & 1\ 0\ 0\ 1 \end{array}$$

$$\begin{array}{r} (-7) \\ +\ (+2) \\ \hline (-5) \end{array} \qquad \begin{array}{rr} & \boxed{0\ 0}\,0\ 0\ 0 \\ & 1\ 0\ 0\ 1 \\ + & 0\ 0\ 1\ 0 \\ \hline & 1\ 0\ 1\ 1 \end{array}$$

$c_4 = 0$
$c_3 = 0$

$c_4 = 1$
$c_3 = 1$

$$\begin{array}{r} (+7) \\ +\ (-2) \\ \hline (+5) \end{array} \qquad \begin{array}{rr} & \boxed{1\ 1}\,1\ 0\ 0 \\ & 0\ 1\ 1\ 1 \\ + & 1\ 1\ 1\ 0 \\ \hline & 1\ 0\ 1\ 0\ 1 \end{array}$$

$$\begin{array}{r} (-7) \\ +\ (-2) \\ \hline (-9) \end{array} \qquad \begin{array}{rr} & \boxed{1\ 0}\,0\ 0\ 0 \\ & 1\ 0\ 0\ 1 \\ + & 1\ 1\ 1\ 0 \\ \hline & 1\ 0\ 1\ 1\ 1 \end{array}$$

$c_4 = 1$
$c_3 = 0$

$$\text{Overflow} = c_3\overline{c}_4 + \overline{c}_3 c_4$$

Examples of determination of overflow

$$c_4 = 0 \qquad c_3 = 1$$

$$\begin{array}{r} (+7) \\ +\ (+2) \\ \hline (+9) \end{array} \qquad \begin{array}{cr} & \boxed{0\ 1}\,1\ 0\ 0 \\ & 0\ 1\ 1\ 1 \\ + & 0\ 0\ 1\ 0 \\ \hline & 1\ 0\ 0\ 1 \end{array}$$

$$\begin{array}{r} (-7) \\ +\ (+2) \\ \hline (-5) \end{array} \qquad \begin{array}{cr} & \boxed{0\ 0}\,0\ 0\ 0 \\ & 1\ 0\ 0\ 1 \\ + & 0\ 0\ 1\ 0 \\ \hline & 1\ 0\ 1\ 1 \end{array}$$

$$c_4 = 0 \qquad c_3 = 0$$

$$c_4 = 1 \qquad c_3 = 1$$

$$\begin{array}{r} (+7) \\ +\ (-2) \\ \hline (+5) \end{array} \qquad \begin{array}{cr} & \boxed{1\ 1}\,1\ 0\ 0 \\ & 0\ 1\ 1\ 1 \\ + & 1\ 1\ 1\ 0 \\ \hline & 1\ 0\ 1\ 0\ 1 \end{array}$$

$$\begin{array}{r} (-7) \\ +\ (-2) \\ \hline (-9) \end{array} \qquad \begin{array}{cr} & \boxed{1\ 0}\,0\ 0\ 0 \\ & 1\ 0\ 0\ 1 \\ + & 1\ 1\ 1\ 0 \\ \hline & 1\ 0\ 1\ 1\ 1 \end{array}$$

$$c_4 = 1 \qquad c_3 = 0$$

$$\text{Overflow} = \underbrace{c_3\overline{c}_4 + \overline{c}_3 c_4}_{\text{XOR}}$$

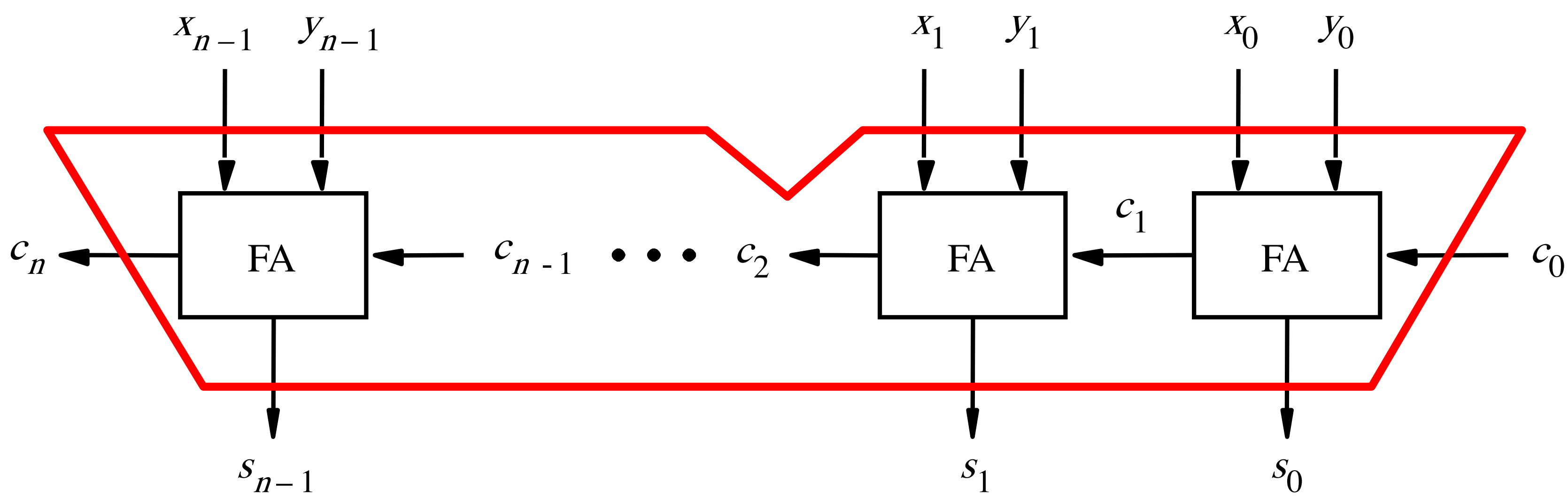
Calculating overflow for 4-bit numbers with only three significant bits

$$
\begin{aligned}
\text{Overflow} &= c_3\overline{c}_4 + \overline{c}_3 c_4 \\
&= c_3 \oplus c_4
\end{aligned}
$$

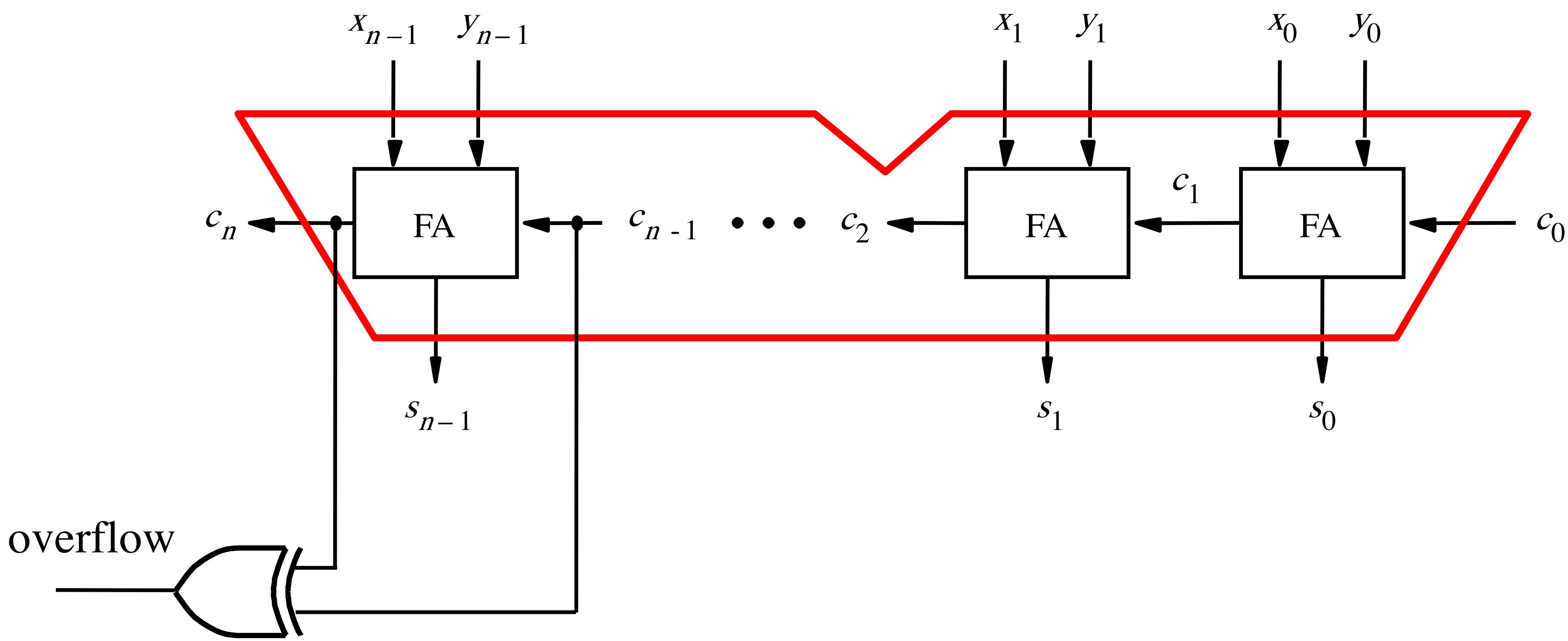
Calculating overflow for n-bit numbers with only n-1 significant bits

$$\text{Overflow} = c_{n-1} \oplus c_n$$

Detecting Overflow



Detecting Overflow (with one extra XOR)



Another way to look at the overflow issue

$$
\begin{array}{rr}
 & X = x_3\ x_2\ x_1\ x_0 \\
+ & Y = y_3\ y_2\ y_1\ y_0 \\
\hline
 & S = s_3\ s_2\ s_1\ s_0
\end{array}
$$

Another way to look at the overflow issue

$$
\begin{array}{rr|c|ccc}
 & X= & x_3 & x_2 & x_1 & x_0 \\
+ & Y= & y_3 & y_2 & y_1 & y_0 \\
\hline
 & S= & s_3 & s_2 & s_1 & s_0
\end{array}
$$

If both numbers that we are adding have the same sign but the sum does not, then we have an overflow.

Examples of determination of overflow

$$\begin{array}{rr} & (+7) \\ + & (+2) \\ \hline & (+9) \end{array} \qquad \begin{array}{rr} & 0\,1\,1\,1 \\ + & 0\,0\,1\,0 \\ \hline & 1\,0\,0\,1 \end{array} \qquad\qquad \begin{array}{rr} & (-7) \\ + & (+2) \\ \hline & (-5) \end{array} \qquad \begin{array}{rr} & 1\,0\,0\,1 \\ + & 0\,0\,1\,0 \\ \hline & 1\,0\,1\,1 \end{array}$$

$$\begin{array}{rr} & (+7) \\ + & (-2) \\ \hline & (+5) \end{array} \qquad \begin{array}{rr} & 0\,1\,1\,1 \\ + & 1\,1\,1\,0 \\ \hline & 1\,0\,1\,0\,1 \end{array} \qquad\qquad \begin{array}{rr} & (-7) \\ + & (-2) \\ \hline & (-9) \end{array} \qquad \begin{array}{rr} & 1\,0\,0\,1 \\ + & 1\,1\,1\,0 \\ \hline & 1\,0\,1\,1\,1 \end{array}$$

Examples of determination of overflow

$$\begin{array}{r} (+7) \\ +\ (+2) \\ \hline (+9) \end{array} \qquad \begin{array}{r} 0\,1\,1\,1 \\ +\quad 0\,0\,1\,0 \\ \hline 1\,0\,0\,1 \end{array} \qquad\qquad \begin{array}{r} (-7) \\ +\ (+2) \\ \hline (-5) \end{array} \qquad \begin{array}{r} 1\,0\,0\,1 \\ +\quad 0\,0\,1\,0 \\ \hline 1\,0\,1\,1 \end{array}$$

$$\begin{array}{r} (+7) \\ +\ (-2) \\ \hline (+5) \end{array} \qquad \begin{array}{r} 0\,1\,1\,1 \\ +\quad 1\,1\,1\,0 \\ \hline 1\,0\,1\,0\,1 \end{array} \qquad\qquad \begin{array}{r} (-7) \\ +\ (-2) \\ \hline (-9) \end{array} \qquad \begin{array}{r} 1\,0\,0\,1 \\ +\quad 1\,1\,1\,0 \\ \hline 1\,0\,1\,1\,1 \end{array}$$

Examples of determination of overflow

$$
\begin{aligned}
x_3 &= 0 \\
y_3 &= 0 \\
s_3 &= 1
\end{aligned}
\qquad
\begin{array}{r}
(+7) \\
+\ (+2) \\ \hline
(+9)
\end{array}
\qquad
\begin{array}{rr}
 & 0\,1\,1\,1 \\
+ & 0\,0\,1\,0 \\ \hline
 & 1\,0\,0\,1
\end{array}
$$

$$
\begin{array}{r}
(-7) \\
+\ (+2) \\ \hline
(-5)
\end{array}
\qquad
\begin{array}{rr}
 & 1\,0\,0\,1 \\
+ & 0\,0\,1\,0 \\ \hline
 & 1\,0\,1\,1
\end{array}
\qquad
\begin{aligned}
x_3 &= 1 \\
y_3 &= 0 \\
s_3 &= 1
\end{aligned}
$$

$$
\begin{aligned}
x_3 &= 0 \\
y_3 &= 1 \\
s_3 &= 0
\end{aligned}
\qquad
\begin{array}{r}
(+7) \\
+\ (-2) \\ \hline
(+5)
\end{array}
\qquad
\begin{array}{rr}
 & 0\,1\,1\,1 \\
+ & 1\,1\,1\,0 \\ \hline
 & 1\,0\,1\,0\,1
\end{array}
$$

$$
\begin{array}{r}
(-7) \\
+\ (-2) \\ \hline
(-9)
\end{array}
\qquad
\begin{array}{rr}
 & 1\,0\,0\,1 \\
+ & 1\,1\,1\,0 \\ \hline
 & 1\,0\,1\,1\,1
\end{array}
\qquad
\begin{aligned}
x_3 &= 1 \\
y_3 &= 1 \\
s_3 &= 0
\end{aligned}
$$

Examples of determination of overflow

$x_3 = 0$
$y_3 = 0$
$s_3 = 1$

$$\begin{array}{rr} & (+7) \\ + & (+2) \\ \hline & (+9) \end{array} \qquad \begin{array}{rcccc} & 0 & 1 & 1 & 1 \\ + & 0 & 0 & 1 & 0 \\ \hline & 1 & 0 & 0 & 1 \end{array}$$

$$\begin{array}{rr} & (-7) \\ + & (+2) \\ \hline & (-5) \end{array} \qquad \begin{array}{rcccc} & 1 & 0 & 0 & 1 \\ + & 0 & 0 & 1 & 0 \\ \hline & 1 & 0 & 1 & 1 \end{array}$$

$x_3 = 1$
$y_3 = 0$
$s_3 = 1$

$x_3 = 0$
$y_3 = 1$
$s_3 = 0$

$$\begin{array}{rr} & (+7) \\ + & (-2) \\ \hline & (+5) \end{array} \qquad \begin{array}{rccccc} & & 0 & 1 & 1 & 1 \\ + & & 1 & 1 & 1 & 0 \\ \hline & 1 & 0 & 1 & 0 & 1 \end{array}$$

$$\begin{array}{rr} & (-7) \\ + & (-2) \\ \hline & (-9) \end{array} \qquad \begin{array}{rccccc} & & 1 & 0 & 0 & 1 \\ + & & 1 & 1 & 1 & 0 \\ \hline & 1 & 0 & 1 & 1 & 1 \end{array}$$

$x_3 = 1$
$y_3 = 1$
$s_3 = 0$

In 2's complement, both +9 and -9 are not representable with 4 bits.

Examples of determination of overflow

$$x_3 = 0 \qquad y_3 = 0 \qquad s_3 = 1$$

$$\begin{array}{r} (+7) \\ +\ (+2) \\ \hline (+9) \end{array} \qquad \begin{array}{rrrrr} & 0 & 1 & 1 & 1 \\ + & 0 & 0 & 1 & 0 \\ \hline & 1 & 0 & 0 & 1 \end{array}$$

$$\begin{array}{r} (-7) \\ +\ (+2) \\ \hline (-5) \end{array} \qquad \begin{array}{rrrrr} & 1 & 0 & 0 & 1 \\ + & 0 & 0 & 1 & 0 \\ \hline & 1 & 0 & 1 & 1 \end{array}$$

$$x_3 = 1 \qquad y_3 = 0 \qquad s_3 = 1$$

$$x_3 = 0 \qquad y_3 = 1 \qquad s_3 = 0$$

$$\begin{array}{r} (+7) \\ +\ (-2) \\ \hline (+5) \end{array} \qquad \begin{array}{rrrrr} & 0 & 1 & 1 & 1 \\ + & 1 & 1 & 1 & 0 \\ \hline 1 & 0 & 1 & 0 & 1 \end{array}$$

$$\begin{array}{r} (-7) \\ +\ (-2) \\ \hline (-9) \end{array} \qquad \begin{array}{rrrrr} & 1 & 0 & 0 & 1 \\ + & 1 & 1 & 1 & 0 \\ \hline 1 & 0 & 1 & 1 & 1 \end{array}$$

$$x_3 = 1 \qquad y_3 = 1 \qquad s_3 = 0$$

Overflow occurs only in these two cases.

Examples of determination of overflow

$x_3 = 0$
$y_3 = 0$
$s_3 = 1$

$$\begin{array}{r} (+7) \\ +\ (+2) \\ \hline (+9) \end{array} \qquad \begin{array}{r} 0\,1\,1\,1 \\ +\ \ 0\,0\,1\,0 \\ \hline 1\,0\,0\,1 \end{array}$$

$x_3 = 1$
$y_3 = 0$
$s_3 = 1$

$$\begin{array}{r} (-7) \\ +\ (+2) \\ \hline (-5) \end{array} \qquad \begin{array}{r} 1\,0\,0\,1 \\ +\ \ 0\,0\,1\,0 \\ \hline 1\,0\,1\,1 \end{array}$$

$x_3 = 0$
$y_3 = 1$
$s_3 = 0$

$$\begin{array}{r} (+7) \\ +\ (-2) \\ \hline (+5) \end{array} \qquad \begin{array}{r} 0\,1\,1\,1 \\ +\ \ 1\,1\,1\,0 \\ \hline 1\,0\,1\,0\,1 \end{array}$$

$x_3 = 1$
$y_3 = 1$
$s_3 = 0$

$$\begin{array}{r} (-7) \\ +\ (-2) \\ \hline (-9) \end{array} \qquad \begin{array}{r} 1\,0\,0\,1 \\ +\ \ 1\,1\,1\,0 \\ \hline 1\,0\,1\,1\,1 \end{array}$$

$$\text{Overflow} = \overline{x}_3\,\overline{y}_3\,s_3 + x_3\,y_3\,\overline{s}_3$$

Another way to look at the overflow issue

$$
\begin{array}{rr|c|ccc}
 & X= & x_3 & x_2 & x_1 & x_0 \\
+ & Y= & y_3 & y_2 & y_1 & y_0 \\
\hline
 & S= & s_3 & s_2 & s_1 & s_0
\end{array}
$$

If both numbers that we are adding have the same sign but the sum does not, then we have an overflow.

$$\text{Overflow} = \overline{x}_3\,\overline{y}_3\,s_3 + x_3\,y_3\,\overline{s}_3$$

Questions?

THE END