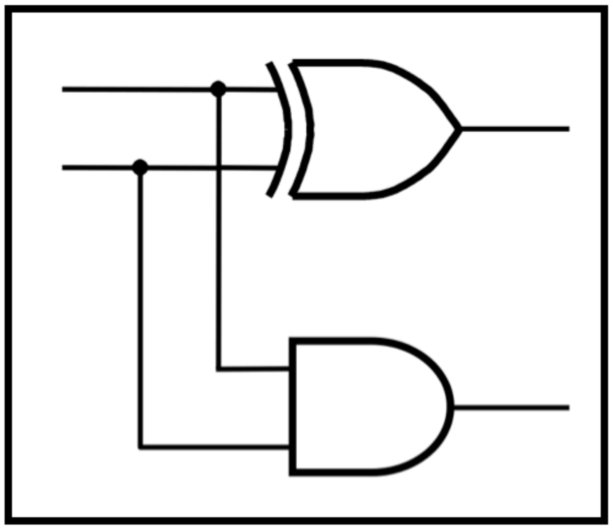


CprE 281: Digital Logic

Instructor: Alexander Stoytchev

http://www.ece.iastate.edu/~alexs/classes/

Fast Adders

CprE 281: Digital Logic
Iowa State University, Ames, IA

Administrative Stuff

- No HW is due next Monday
- HW 6 will be due on Monday Oct. 5.

Administrative Stuff

- Labs next week
- Mini-Project
- This is worth 4% of your grade (x2 labs)
- https://www.ece.iastate.edu/~alexs/classes/ 2020_Fall_281/labs/Mini_Project/ /

Quick Review

The problems in which row are easier to calculate?

$$\begin{array}{r} 82 \\ -\ 61 \\ \hline ?? \end{array} \qquad \begin{array}{r} 48 \\ -\ 26 \\ \hline ?? \end{array} \qquad \begin{array}{r} 32 \\ -\ 11 \\ \hline ?? \end{array}$$

$$\begin{array}{r} 82 \\ -\ 64 \\ \hline ?? \end{array} \qquad \begin{array}{r} 48 \\ -\ 29 \\ \hline ?? \end{array} \qquad \begin{array}{r} 32 \\ -\ 13 \\ \hline ?? \end{array}$$

The problems in which row are easier to calculate?

$$\begin{array}{r} 82 \\ -\ 61 \\ \hline 21 \end{array} \qquad \begin{array}{r} 48 \\ -\ 26 \\ \hline 22 \end{array} \qquad \begin{array}{r} 32 \\ -\ 11 \\ \hline 21 \end{array}$$

Why?

$$\begin{array}{r} 82 \\ -\ 64 \\ \hline 18 \end{array} \qquad \begin{array}{r} 48 \\ -\ 29 \\ \hline 19 \end{array} \qquad \begin{array}{r} 32 \\ -\ 13 \\ \hline 19 \end{array}$$

Another Way to Do Subtraction

$82 – 64 = 82 + 100 – 100 - 64$

Another Way to Do Subtraction

$$82 - 64 = 82 + 100 - 100 - 64$$

$$= 82 + (100 - 64) - 100$$

Another Way to Do Subtraction

$$
\begin{aligned}
82 - 64 &= 82 + 100 - 100 - 64 \\
&= 82 + (100 - 64) - 100 \\
&= 82 + (99 + 1 - 64) - 100
\end{aligned}
$$

Another Way to Do Subtraction

$$82 – 64 = 82 + 100 – 100 - 64$$

$$= 82 + (100 – 64) - 100$$

$$= 82 + (99 + 1 – 64) - 100$$

$$= 82 + (99 – 64) +1 - 100$$

Another Way to Do Subtraction

$$82 - 64 = 82 + 100 - 100 - 64$$

$$= 82 + (100 - 64) - 100$$

$$= 82 + (99 + 1 - 64) - 100$$

Does not require borrows

$$= 82 + (99 - 64) + 1 - 100$$

9’s Complement (subtract each digit from 9)

$$\begin{array}{r} 99 \\ -\ 64 \\ \hline 35 \end{array}$$

10’s Complement
(subtract each digit from 9 and add 1 to the result)

$$\begin{array}{r} 99 \\ -\ 64 \\ \hline 35 \end{array} + 1 = 36$$

Another Way to Do Subtraction

$$82 - 64 = 82 + (99 - 64) + 1 - 100$$

Another Way to Do Subtraction

$$82 - 64 = 82 + \overbrace{(99 - 64)}^{\text{9's complement}} + 1 - 100$$

Another Way to Do Subtraction

$$82 - 64 = 82 + \overset{\text{9's complement}}{(99 - 64)} + 1 - 100$$

$$= 82 + 35 + 1 - 100$$

Another Way to Do Subtraction

9's complement

$$82 - 64 = 82 + (99 - 64) + 1 - 100$$

10's complement

$$= 82 + 35 + 1 - 100$$

Another Way to Do Subtraction

9's complement

$$82 - 64 = 82 + (99 - 64) + 1 - 100$$

10's complement

$$= 82 + 35 + 1 - 100$$

$$= 82 + 36 - 100$$

Another Way to Do Subtraction

9's complement

$$82 - 64 = 82 + (99 - 64) + 1 - 100$$

10's complement

$$= 82 + 35 + 1 - 100$$

$$= 82 + 36 - 100$$ // Add the first two.

$$= 118 - 100$$

Another Way to Do Subtraction

9's complement

$$82 - 64 = 82 + (99 - 64) + 1 - 100$$

10's complement

$$= 82 + 35 + 1 - 100$$

$$= 82 + 36 - 100$$ // Add the first two.

$$= 118 - 100$$ // Just delete the leading 1.
// No need to subtract 100.

$$= 18$$

1’s Complement

1’s complement
(subtract each digit from 1)

Let K be the negative equivalent of an n-bit positive number P.

Then, in 1’s complement representation K is obtained by subtracting P from $2^n - 1$, namely

$$K = (2^n - 1) - P$$

This means that K can be obtained by inverting all bits of P.

1's complement
(subtract each digit from 1)

Let K be the negative equivalent of an 8-bit positive number P.

Then, in 1's complement representation K is obtained by subtracting P from $2^8 - 1$, namely

$$K = (2^8 - 1) - P = 255 - P$$

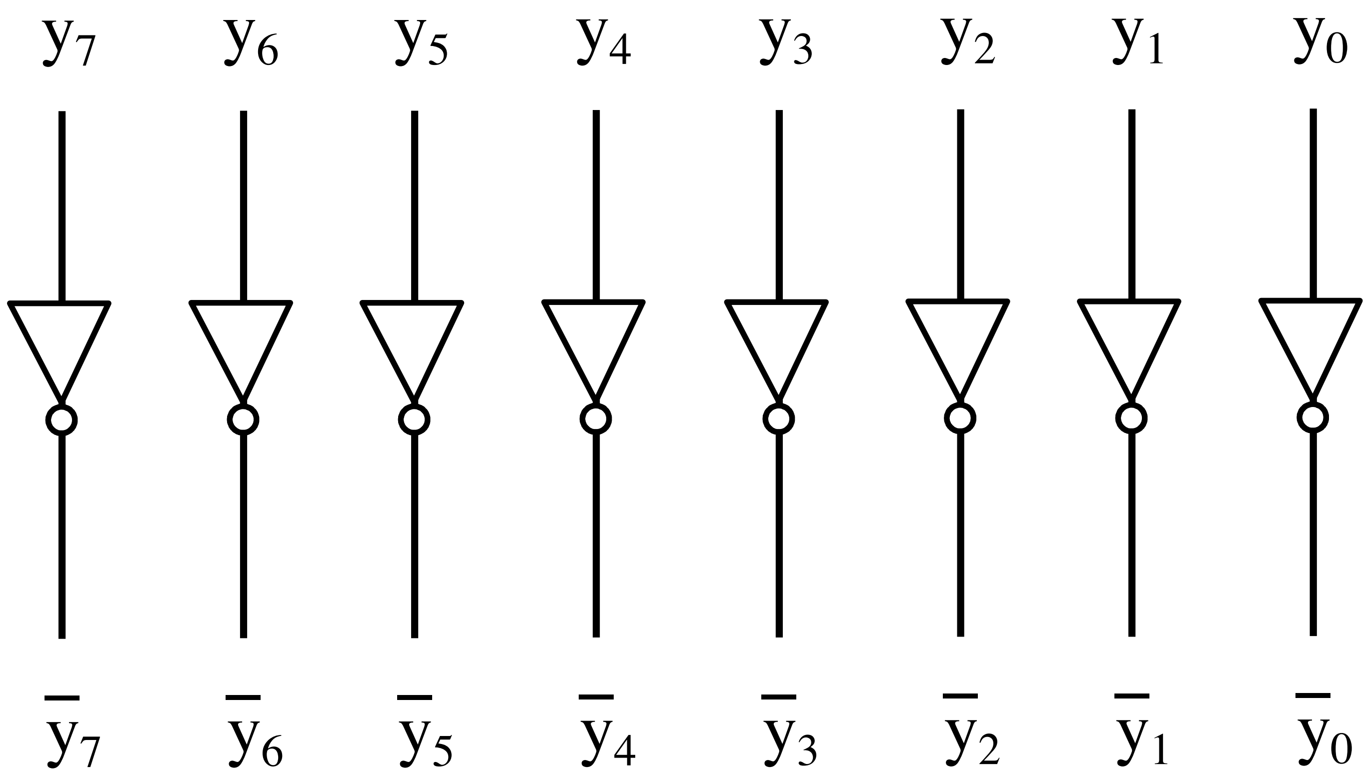
This means that K can be obtained by inverting all bits of P.

Provided that P is between 0 and 127, because the most significant bit must be zero to indicate that it is positive.

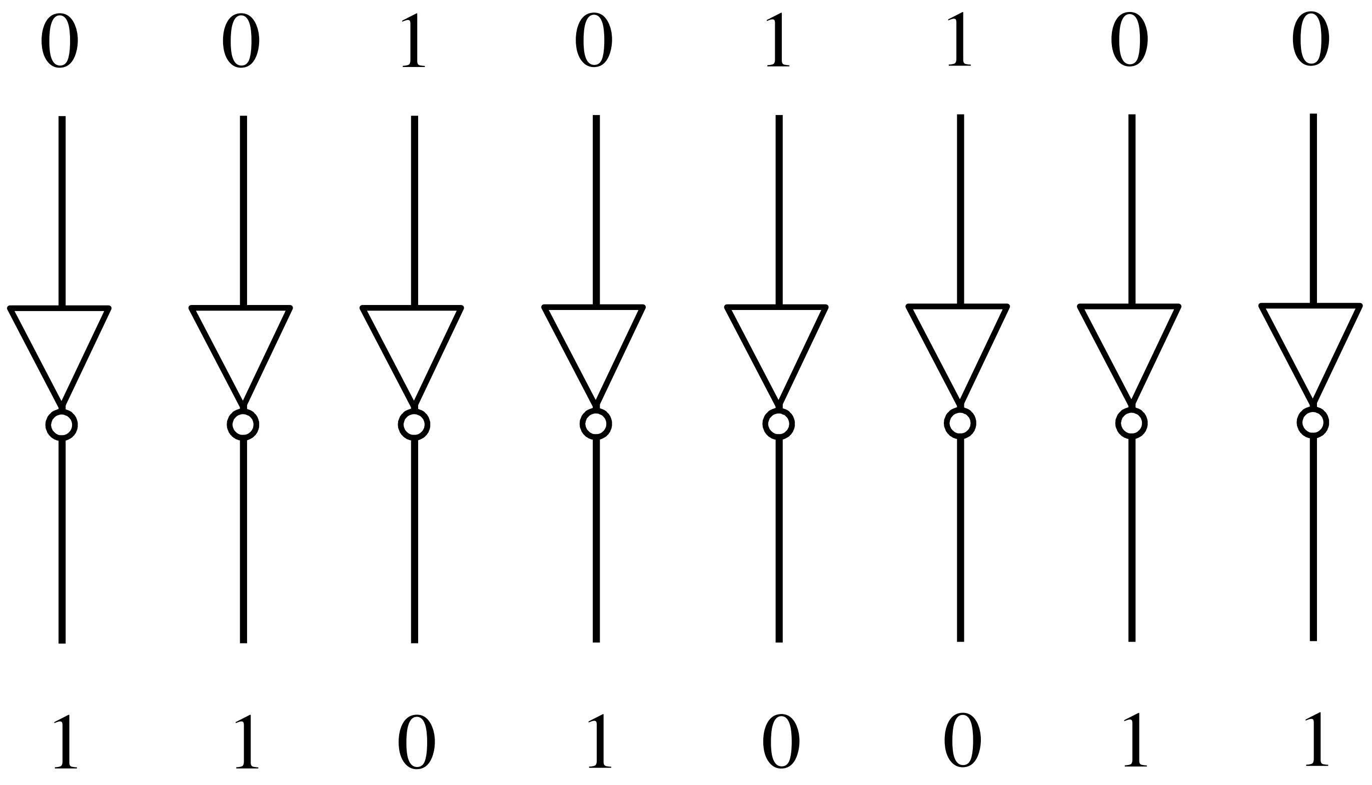
1's complement
(subtract each digit from 1)

$$
\begin{array}{r}
\; 1\;\;1\;\;1\;\;1\;\;1\;\;1\;\;1\;\;1 \\
-\; 0\;\;0\;\;1\;\;0\;\;1\;\;1\;\;0\;\;0 \\
\hline
1\;\;1\;\;0\;\;1\;\;0\;\;0\;\;1\;\;1
\end{array}
$$

Circuit for negating a number stored in 1's complement representation



Circuit for negating a number stored in 1's complement representation



2’s Complement

2’s complement

Let K be the negative equivalent of an n-bit positive number P.

Then, in 2’s complement representation K is obtained by subtracting P from 2^n, namely

$$K = 2^n - P$$

Deriving 2’s complement

For a positive n-bit number P, let K_1 and K_2 denote its 1’s and 2’s complements, respectively.

$$K_1 = (2^n - 1) - P$$

$$K_2 = 2^n - P$$

Since $K_2 = K_1 + 1$, it is evident that in a logic circuit the 2’s complement can computed by inverting all bits of P and then adding 1 to the resulting 1’s-complement number.

Deriving 2’s complement

For a positive 8-bit number P, let K_1 and K_2 denote its 1’s and 2’s complements, respectively.

$$K_1 = (2^n - 1) - P = 255 - P$$

$$K_2 = 2^n - P = 256 - P$$

Since $K_2 = K_1 + 1$, it is evident that in a logic circuit the 2’s complement can computed by inverting all bits of P and then adding 1 to the resulting 1’s-complement number.

Find the 2’s complement of …

0 1 0 1

0 0 1 0

0 1 0 0

0 1 1 1

Find the 2's complement of …

0 1 0 1	0 0 1 0
1 0 1 0	1 1 0 1
0 1 0 0	0 1 1 1
1 0 1 1	1 0 0 0

Invert all bits.

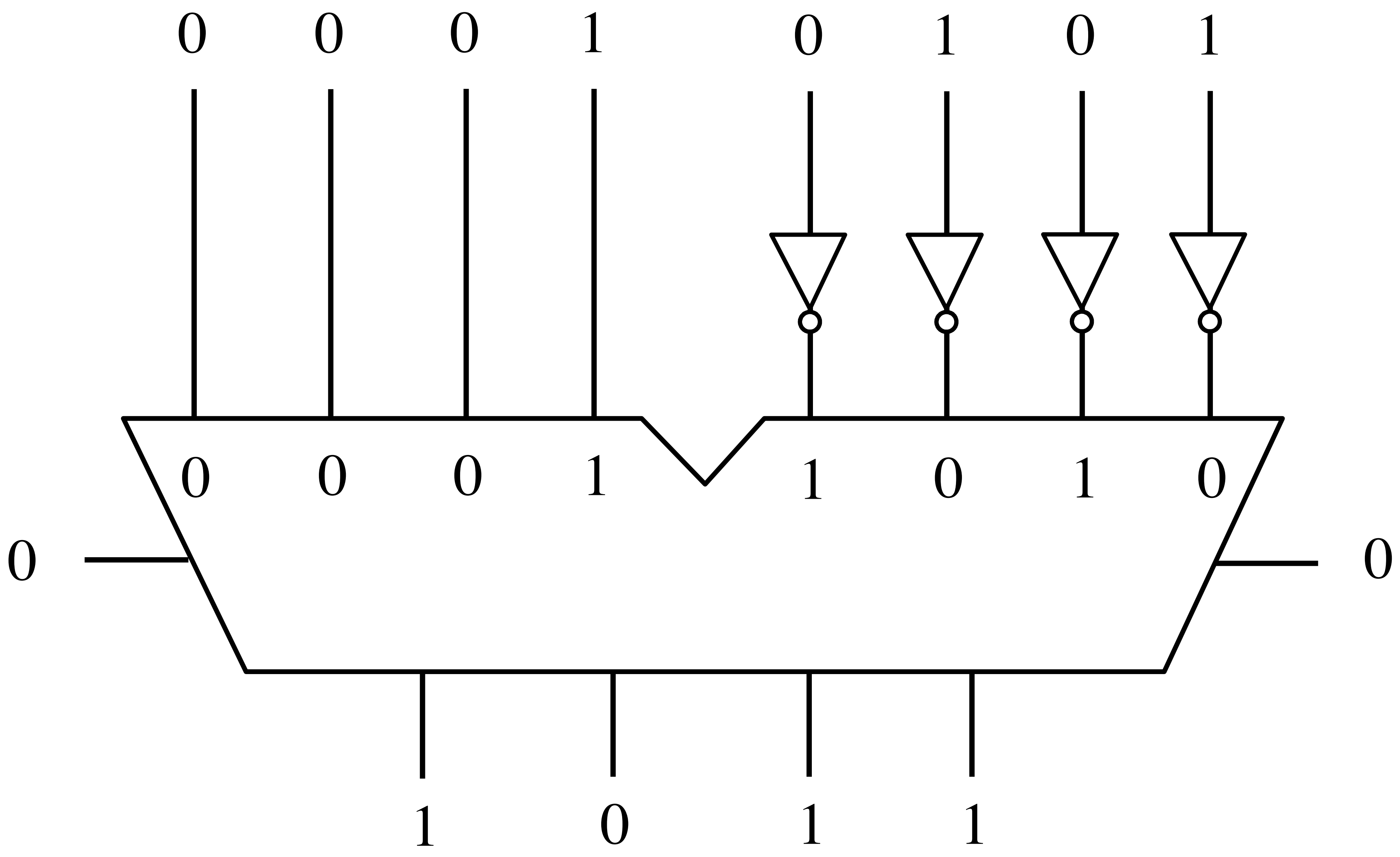
Find the 2's complement of …

$$\begin{array}{r} 0\,1\,0\,1 \\ 1\,0\,1\,0 \\ +\quad 1 \\ \hline 1\,0\,1\,1 \end{array} \qquad \begin{array}{r} 0\,0\,1\,0 \\ 1\,1\,0\,1 \\ +\quad 1 \\ \hline 1\,1\,1\,0 \end{array}$$

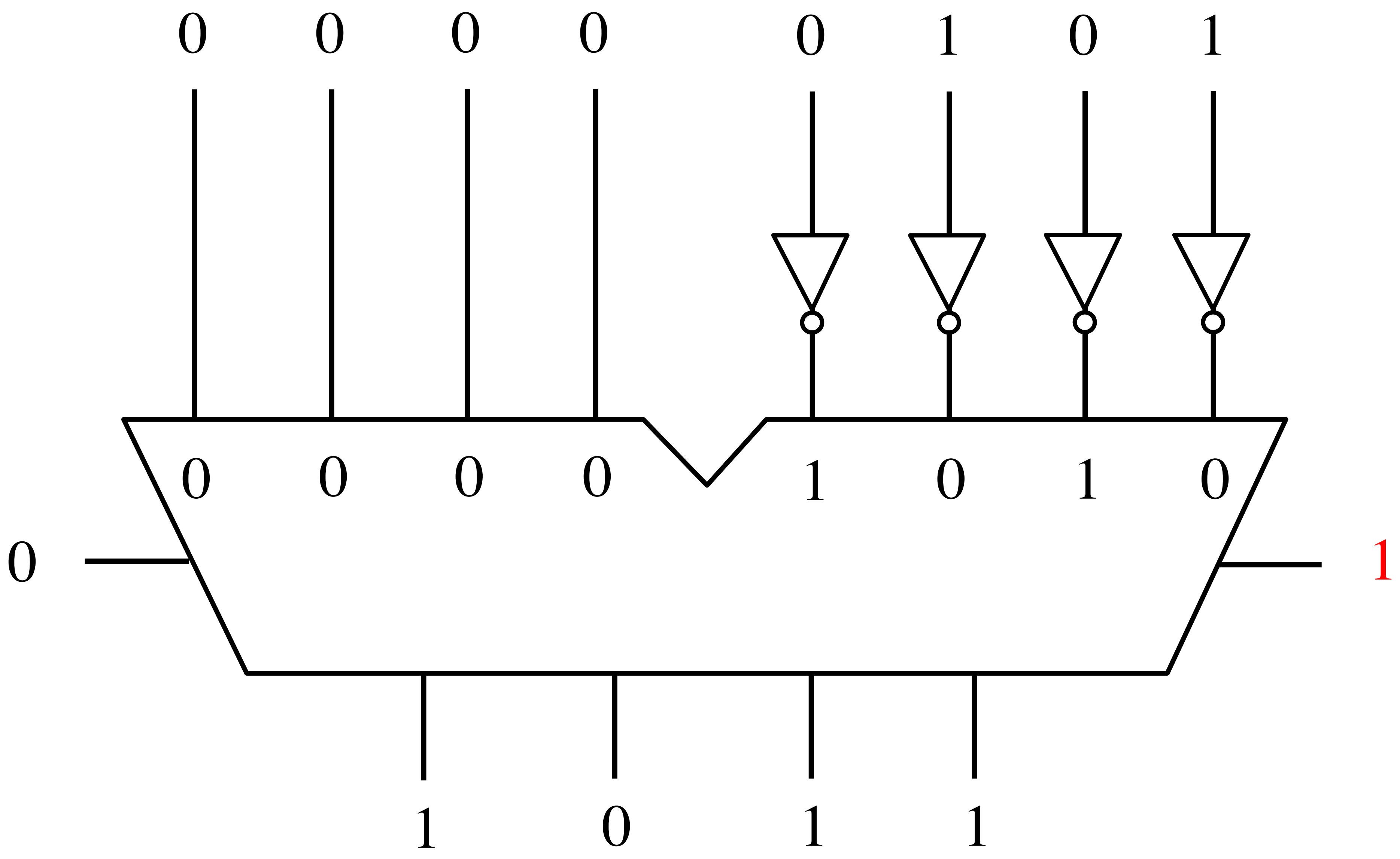
$$\begin{array}{r} 0\,1\,0\,0 \\ 1\,0\,1\,1 \\ +\quad 1 \\ \hline 1\,1\,0\,0 \end{array} \qquad \begin{array}{r} 0\,1\,1\,1 \\ 1\,0\,0\,0 \\ +\quad 1 \\ \hline 1\,0\,0\,1 \end{array}$$

Then add 1.

Circuit for negating a number stored in 2's complement representation



Circuit for negating a number stored in 2's complement representation



Addition of two numbers stored in 2’s complement representation

There are four cases to consider

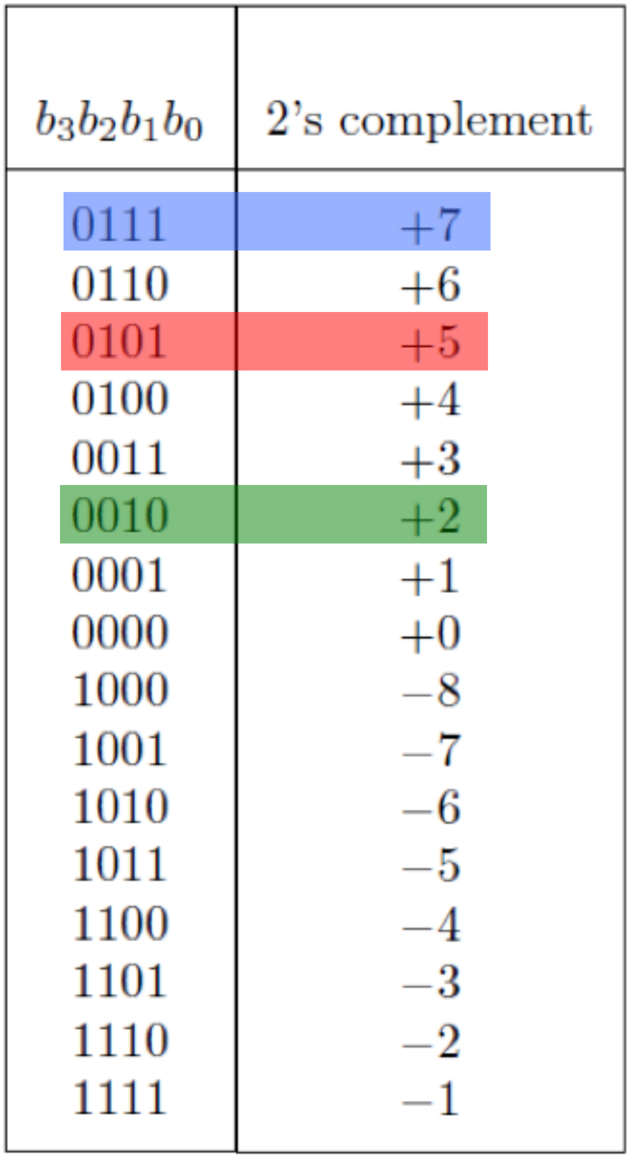
- `(+5)   +   (+2)`
- `(–5)   +   (+2)`
- `(+5)   +   (–2)`
- `(–5)   +   (–2)`

There are four cases to consider

- `(+5)  +  (+2)      positive plus positive`
- `(–5)  +  (+2)      negative plus positive`
- `(+5)  +  (–2)      positive plus negative`
- `(–5)  +  (–2)      negative plus negative`

Positive plus positive

$$\begin{array}{rr} (+5) & 0\,1\,0\,1 \\ +\ (+2) & +\ 0\,0\,1\,0 \\ \hline (+7) & 0\,1\,1\,1 \end{array}$$



$b_3b_2b_1b_0$	2's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−8
1001	−7
1010	−6
1011	−5
1100	−4
1101	−3
1110	−2
1111	−1

[Figure 3.9 from the textbook]

Negative plus positive

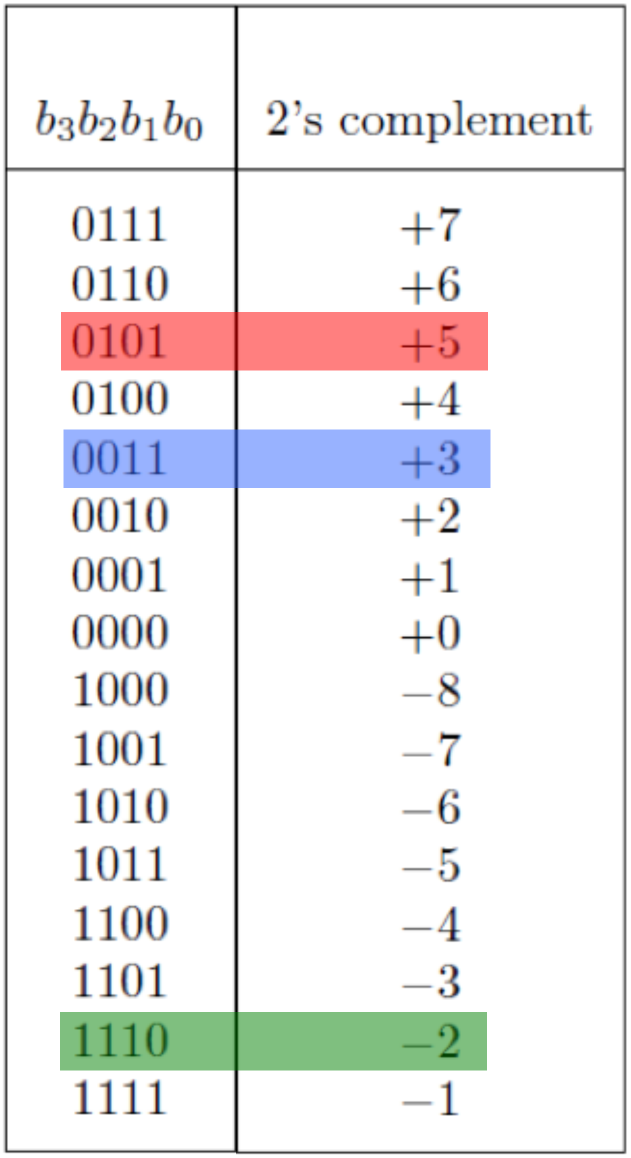
```
  (–5)        1 0 1 1
+ (+ 2)     + 0 0 1 0
-------     ---------
  (–3)        1 1 0 1
```

$b_3b_2b_1b_0$	2's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−8
1001	−7
1010	−6
1011	−5
1100	−4
1101	−3
1110	−2
1111	−1

[Figure 3.9 from the textbook]

Positive plus negative

$$
\begin{array}{rr}
 & (+5) \\
+ & (-2) \\
\hline
 & (+3)
\end{array}
\qquad\qquad
\begin{array}{rr}
 & 0\,1\,0\,1 \\
+ & 1\,1\,1\,0 \\
\hline
\underset{\text{ignore}}{\overset{\uparrow}{1}}\!\!\!\!\! & 0\,0\,1\,1
\end{array}
$$



$b_3b_2b_1b_0$	2's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−8
1001	−7
1010	−6
1011	−5
1100	−4
1101	−3
1110	−2
1111	−1

[Figure 3.9 from the textbook]

Negative plus negative

$$\begin{array}{r} (-5)\\ +\ (-2)\\ \hline (-7)\end{array} \qquad \begin{array}{r} 1\,0\,1\,1\\ +\ 1\,1\,1\,0\\ \hline \underset{\substack{\uparrow\\ \text{ignore}}}{1}\,1\,0\,0\,1\end{array}$$

$b_3b_2b_1b_0$	2's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−8
1001	−7
1010	−6
1011	−5
1100	−4
1101	−3
1110	−2
1111	−1

[Figure 3.9 from the textbook]

Subtraction of two numbers stored in 2’s complement representation

There are four cases to consider

- (+5) – (+2)
- (–5) – (+2)
- (+5) – (–2)
- (–5) – (–2)

There are four cases to consider

- **(+5) – (+2)** **positive minus positive**
- **(–5) – (+2)** **negative minus positive**
- **(+5) – (–2)** **positive minus negative**
- **(–5) – (–2)** **negative minus negative**

There are four cases to consider

- (+5) – (+2)
- (–5) – (+2)
- (+5) – (–2)
- (–5) – (–2)

There are four cases to consider

- $(+5) - (+2) = (+5) + (-2)$
- $(-5) - (+2) = (-5) + (-2)$
- $(+5) - (-2) = (+5) + (+2)$
- $(-5) - (-2) = (-5) + (+2)$

There are four cases to consider

- $(+5) - (+2) = (+5) + (-2)$
- $(-5) - (+2) = (-5) + (-2)$
- $(+5) - (-2) = (+5) + (+2)$
- $(-5) - (-2) = (-5) + (+2)$

We can change subtraction into addition ...

There are four cases to consider

- (+5) – (+2) = (+5) + (–2)
- (–5) – (+2) = (–5) + (–2)
- (+5) – (–2) = (+5) + (+2)
- (–5) – (–2) = (–5) + (+2)

… if we negate the second number.

There are four cases to consider

- (+5) – (+2) = (+5) + (–2)
- (–5) – (+2) = (–5) + (–2)
- (+5) – (–2) = (+5) + (+2)
- (–5) – (–2) = (–5) + (+2)

There are the four addition cases
(arranged in a shuffled order)

Positive minus positive

$$\begin{array}{r} (+5) \\ -\ (+2) \\ \hline (+3) \end{array} \qquad \begin{array}{r} 0\,1\,0\,1 \\ -\ 0\,0\,1\,0 \\ \hline \end{array}$$

$b_3b_2b_1b_0$	2's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−8
1001	−7
1010	−6
1011	−5
1100	−4
1101	−3
1110	−2
1111	−1

[Figure 3.10 from the textbook]

Convert to: Positive plus negative

$$\begin{array}{r} (+5) \\ -\ (+2) \\ \hline (+3) \end{array} \qquad \begin{array}{r} 0\,1\,0\,1 \\ -\ 0\,0\,1\,0 \\ \hline \end{array} \quad \Longrightarrow \quad \begin{array}{r} 0\,1\,0\,1 \\ +\ 1\,1\,1\,0 \\ \hline \underset{\text{ignore}}{\underset{\uparrow}{1}}\,0\,0\,1\,1 \end{array} \qquad \begin{array}{r} (+5) \\ +\ (-2) \\ \hline (+3) \end{array}$$

$b_3b_2b_1b_0$	2's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−8
1001	−7
1010	−6
1011	−5
1100	−4
1101	−3
1110	−2
1111	−1

[Figure 3.10 from the textbook]

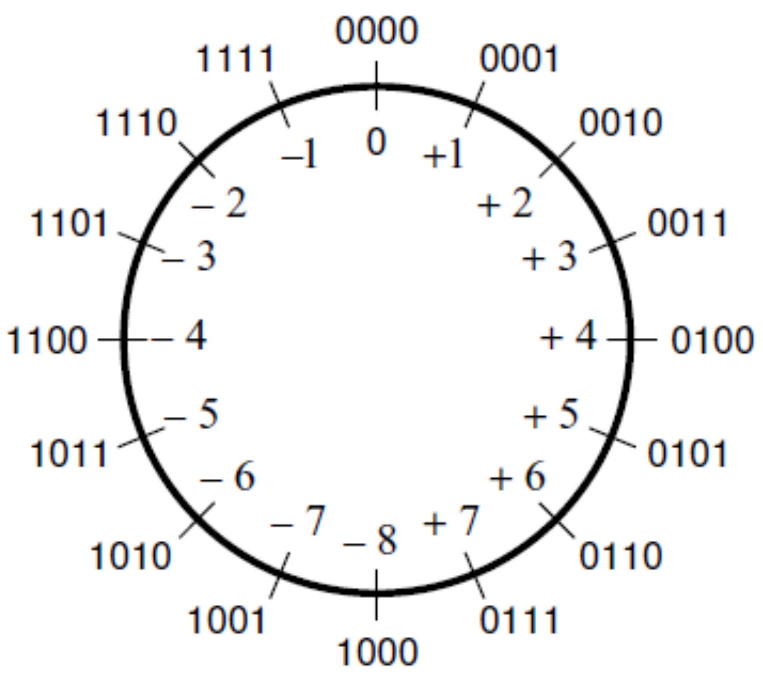
Convert to: Positive plus negative

$$\begin{array}{r} (+5) \\ -\ (+2) \\ \hline (+3) \end{array} \qquad \begin{array}{r} 0\,1\,0\,1 \\ -\ 0\,0\,1\,0 \\ \hline \end{array} \quad \Longrightarrow \quad \begin{array}{r} 0\,1\,0\,1 \\ +\ 1\,1\,1\,0 \\ \hline \underset{\substack{\uparrow \\ \text{ignore}}}{1}\ 0\,0\,1\,1 \end{array} \qquad \begin{array}{r} (+5) \\ +\ (-2) \\ \hline (+3) \end{array}$$

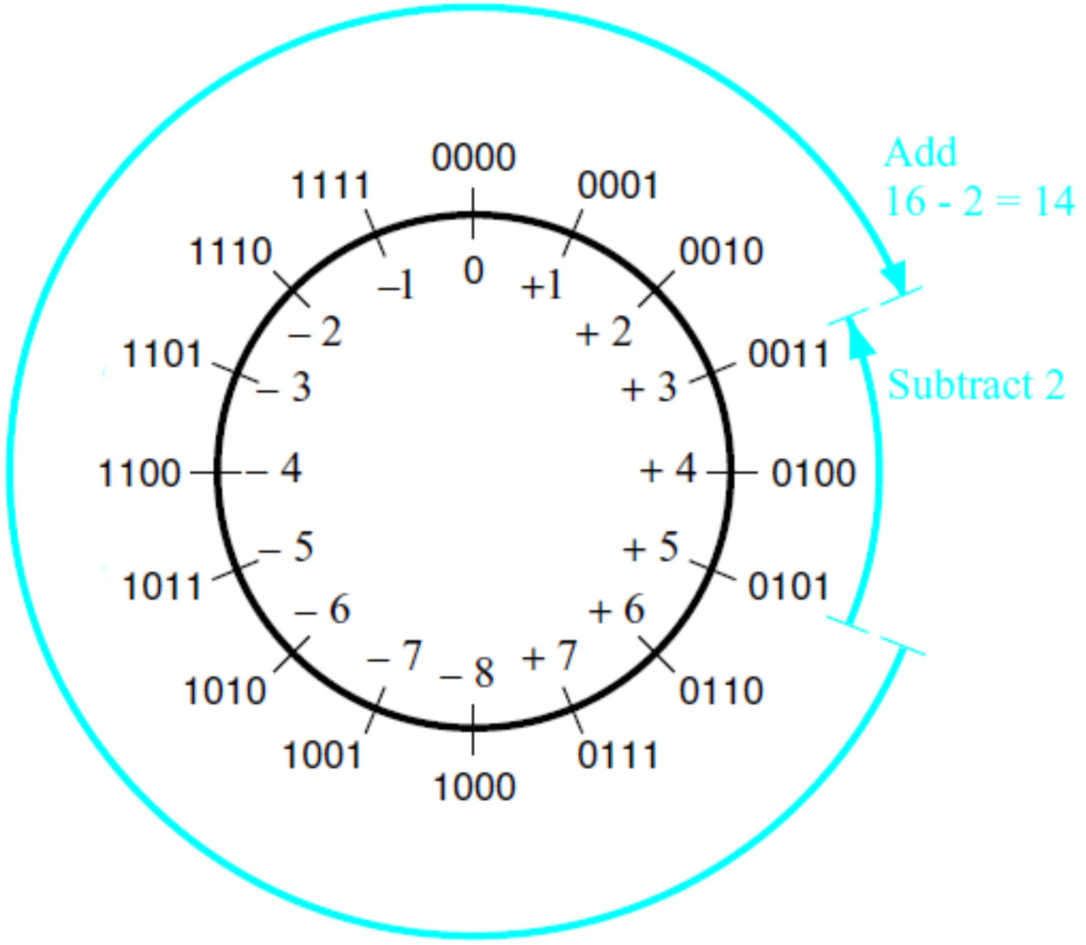
$b_3b_2b_1b_0$	2's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−8
1001	−7
1010	−6
1011	−5
1100	−4
1101	−3
1110	−2
1111	−1

[Figure 3.10 from the textbook]

Graphical interpretation of four-bit 2's complement numbers



(a) The number circle



(b) Subtracting 2 by adding its 2's complement

[Figure 3.11 from the textbook]

Negative minus positive

```
  (−5)        1 0 1 1
− (+2)      − 0 0 1 0
──────      ─────────
  (−7)
```

$b_3b_2b_1b_0$	2's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−8
1001	−7
1010	−6
1011	−5
1100	−4
1101	−3
1110	−2
1111	−1

[Figure 3.10 from the textbook]

Convert to: Negative plus negative

$$\begin{array}{r} (-5) \\ -\ (+2) \\ \hline (-7) \end{array} \qquad \begin{array}{r} 1\,0\,1\,1 \\ -\ 0\,0\,1\,0 \\ \hline \end{array} \quad \Longrightarrow \quad \begin{array}{r} 1\,0\,1\,1 \\ +\ 1\,1\,1\,0 \\ \hline \underset{\substack{\uparrow \\ \text{ignore}}}{1}\,1\,0\,0\,1 \end{array} \qquad \begin{array}{r} (-5) \\ +\ (-2) \\ \hline (-7) \end{array}$$

$b_3b_2b_1b_0$	2's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−8
1001	−7
1010	−6
1011	−5
1100	−4
1101	−3
1110	−2
1111	−1

[Figure 3.10 from the textbook]

Positive minus negative

```
  (+ 5)       0 1 0 1
– (–2)      – 1 1 1 0
-------     ---------
  (+ 7)
```

$b_3b_2b_1b_0$	2's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−8
1001	−7
1010	−6
1011	−5
1100	−4
1101	−3
1110	−2
1111	−1

[Figure 3.10 from the textbook]

Convert to: Positive plus positive

$$\begin{array}{rrcrr} (+5) & 0\,1\,0\,1 & & 0\,1\,0\,1 & (+5) \\ -\ (-2) & -\ 1\,1\,1\,0 & \Longrightarrow & +\ 0\,0\,1\,0 & +\ (+2) \\ \hline (+7) & & & 0\,1\,1\,1 & (+7) \end{array}$$

$b_3b_2b_1b_0$	2's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−8
1001	−7
1010	−6
1011	−5
1100	−4
1101	−3
1110	−2
1111	−1

[Figure 3.10 from the textbook]

Negative minus negatie

```
   (−5)        1 0 1 1
 − (−2)      − 1 1 1 0
 ------      ---------
   (−3)
```

$b_3b_2b_1b_0$	2's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−8
1001	−7
1010	−6
1011	−5
1100	−4
1101	−3
1110	−2
1111	−1

[Figure 3.10 from the textbook]

Convert to: Negative plus positive

$$\begin{array}{r} (-5) \\ -\ (-2) \\ \hline (-3) \end{array} \qquad \begin{array}{r} 1\,0\,1\,1 \\ -\ 1\,1\,1\,0 \\ \hline \end{array} \quad \Longrightarrow \quad \begin{array}{r} 1\,0\,1\,1 \\ +\ 0\,0\,1\,0 \\ \hline 1\,1\,0\,1 \end{array} \qquad \begin{array}{r} (-5) \\ +\ (+2) \\ \hline (-3) \end{array}$$

$b_3b_2b_1b_0$	2's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	−8
1001	−7
1010	−6
1011	−5
1100	−4
1101	−3
1110	−2
1111	−1

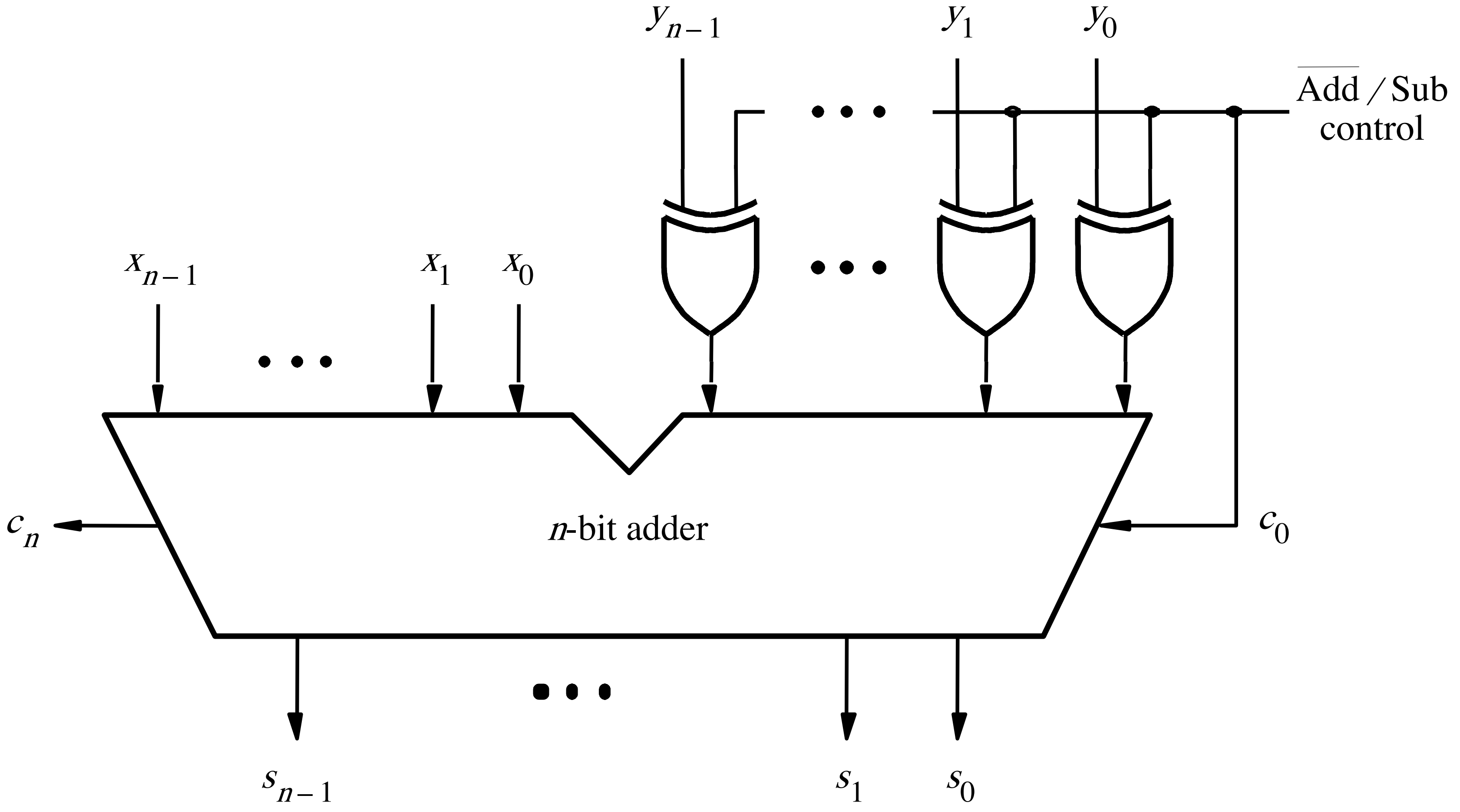
[Figure 3.10 from the textbook]

Take Home Message

- **Subtraction can be performed by simply negating the second number and adding it to the first, regardless of the signs of the two numbers.**

- **Thus, the same adder circuit can be used to perform <u>both addition and subtraction</u> !!!**

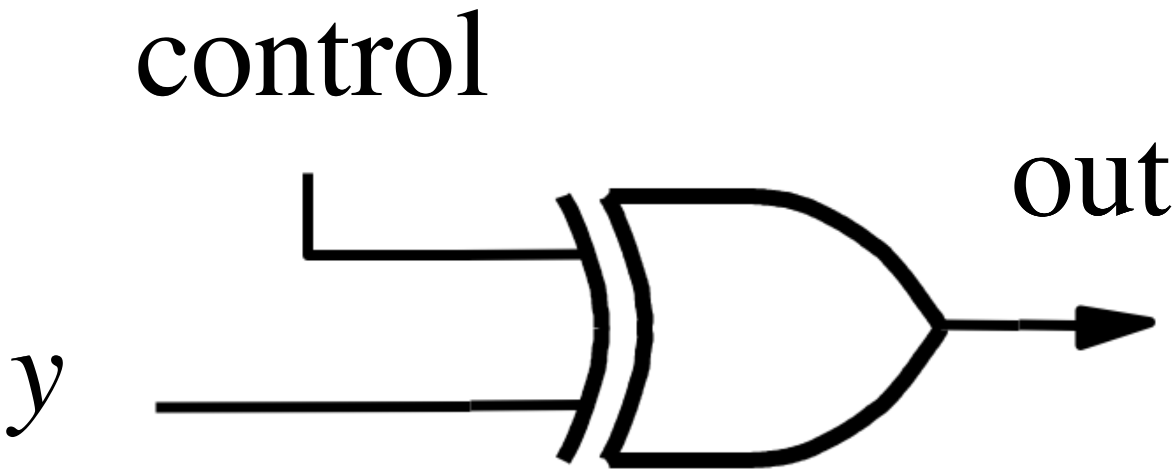
Adder/subtractor unit



[Figure 3.12 from the textbook]

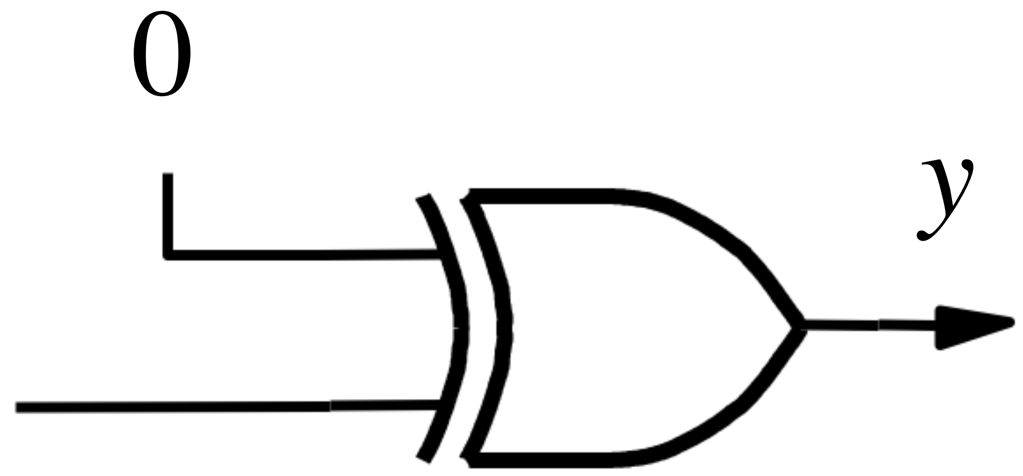
XOR Tricks

control	*y*	out
0	0	0
0	1	1
1	0	1
1	1	0



XOR as a repeater

control	y	out
0	0	0
0	1	1

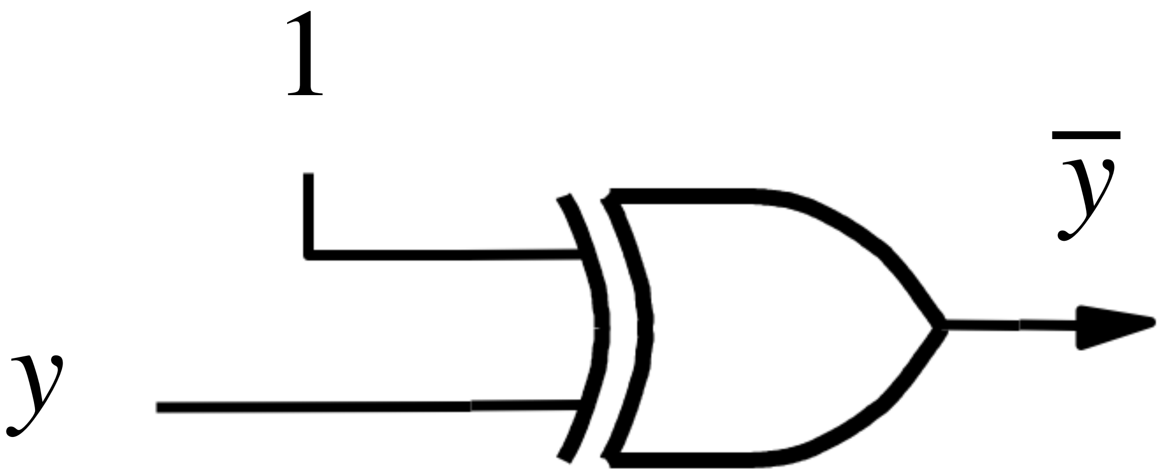


XOR as a repeater

control	y	out
0	0	0
0	1	1

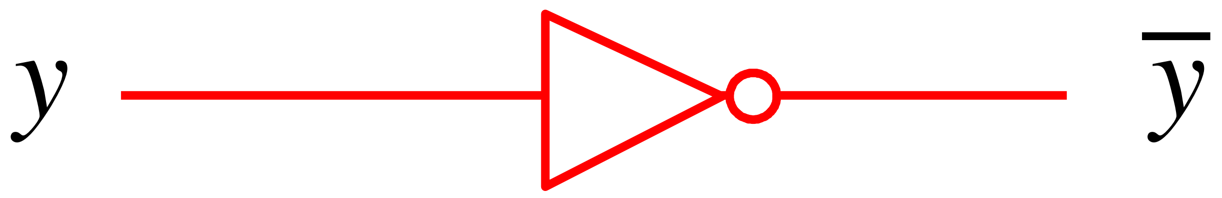
XOR as an inverter

control	y	out
▨	▨	▨
1	0	1
1	1	0

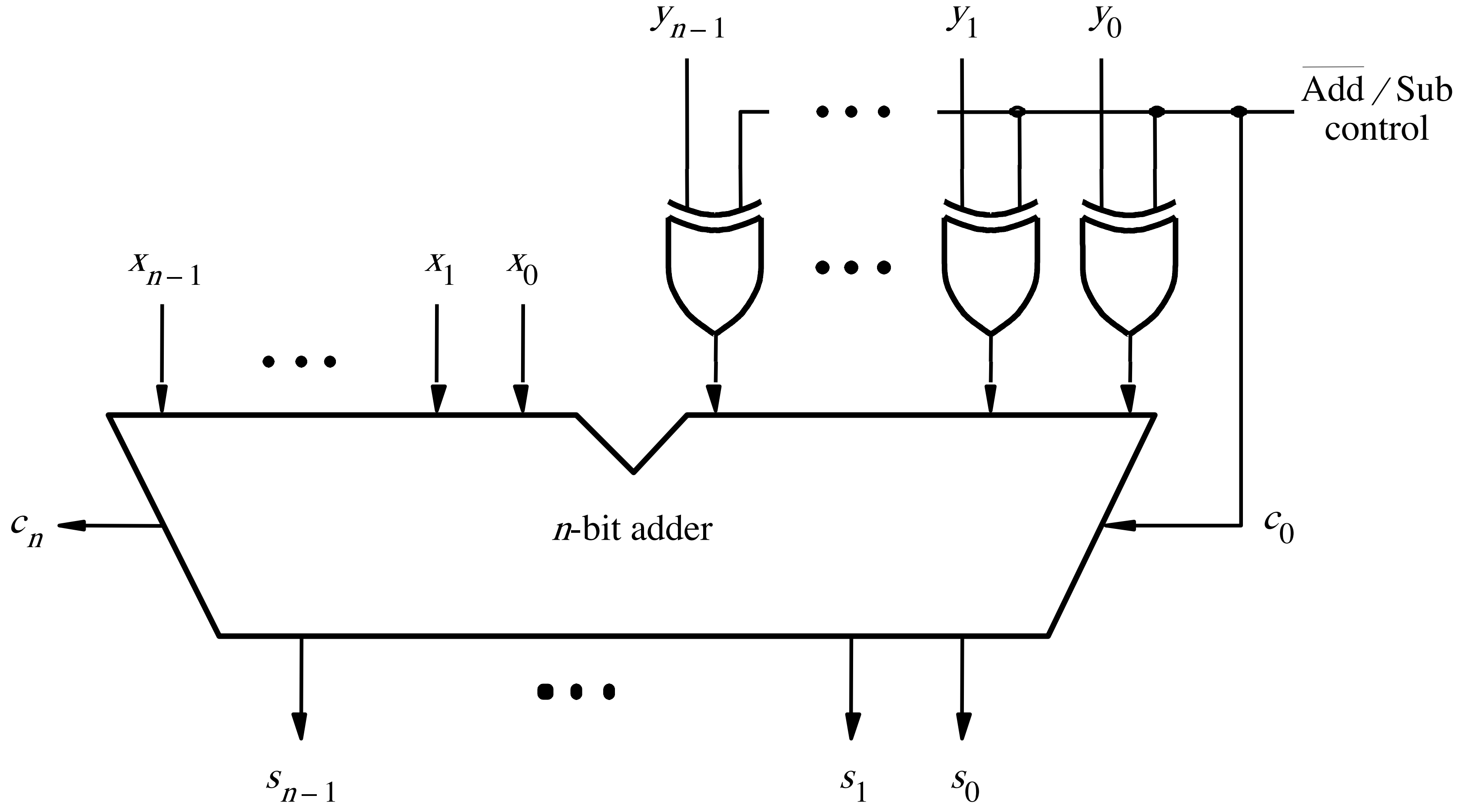


XOR as an inverter

control	y	out
1	0	1
1	1	0

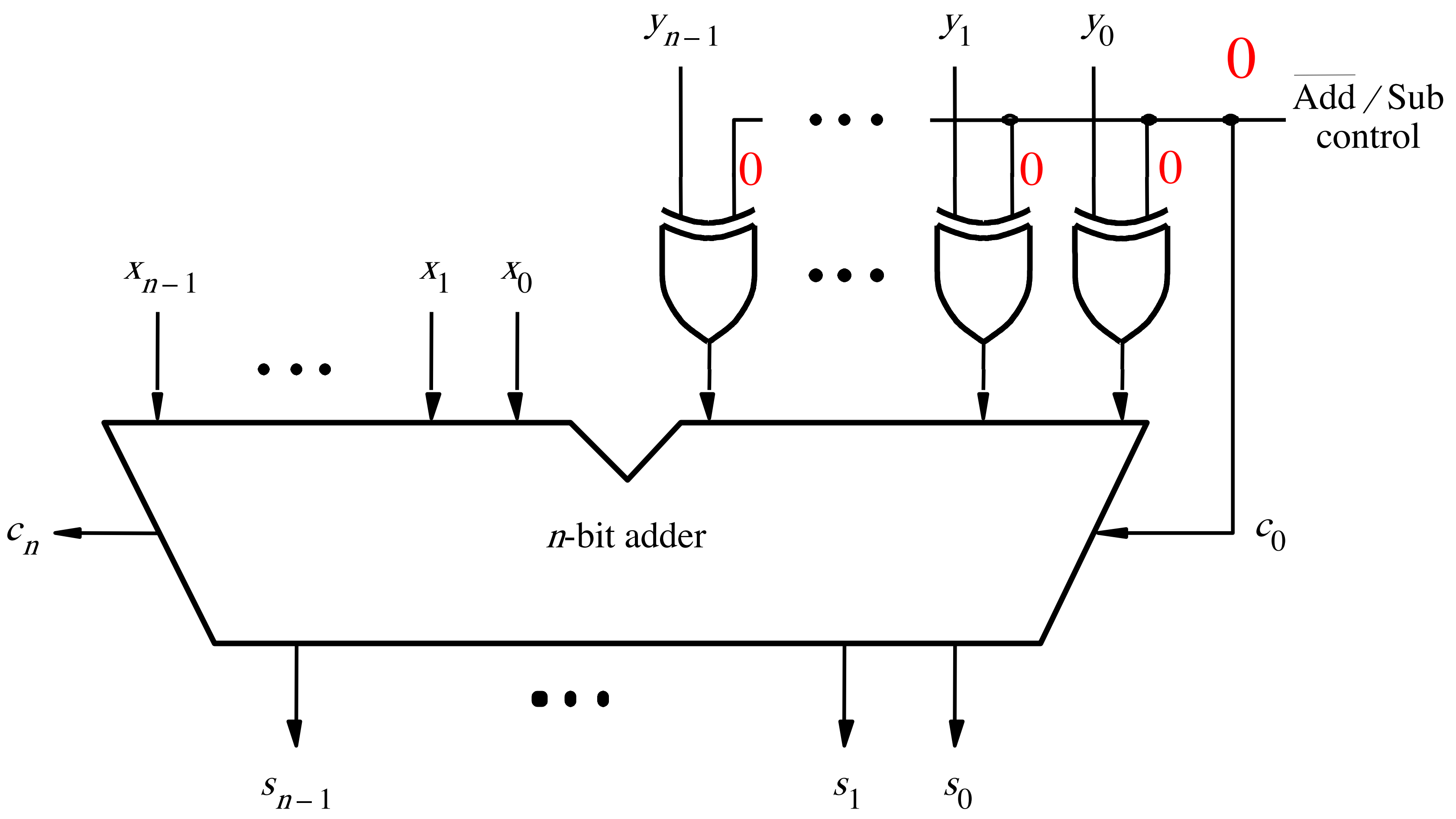


Addition: when control = 0



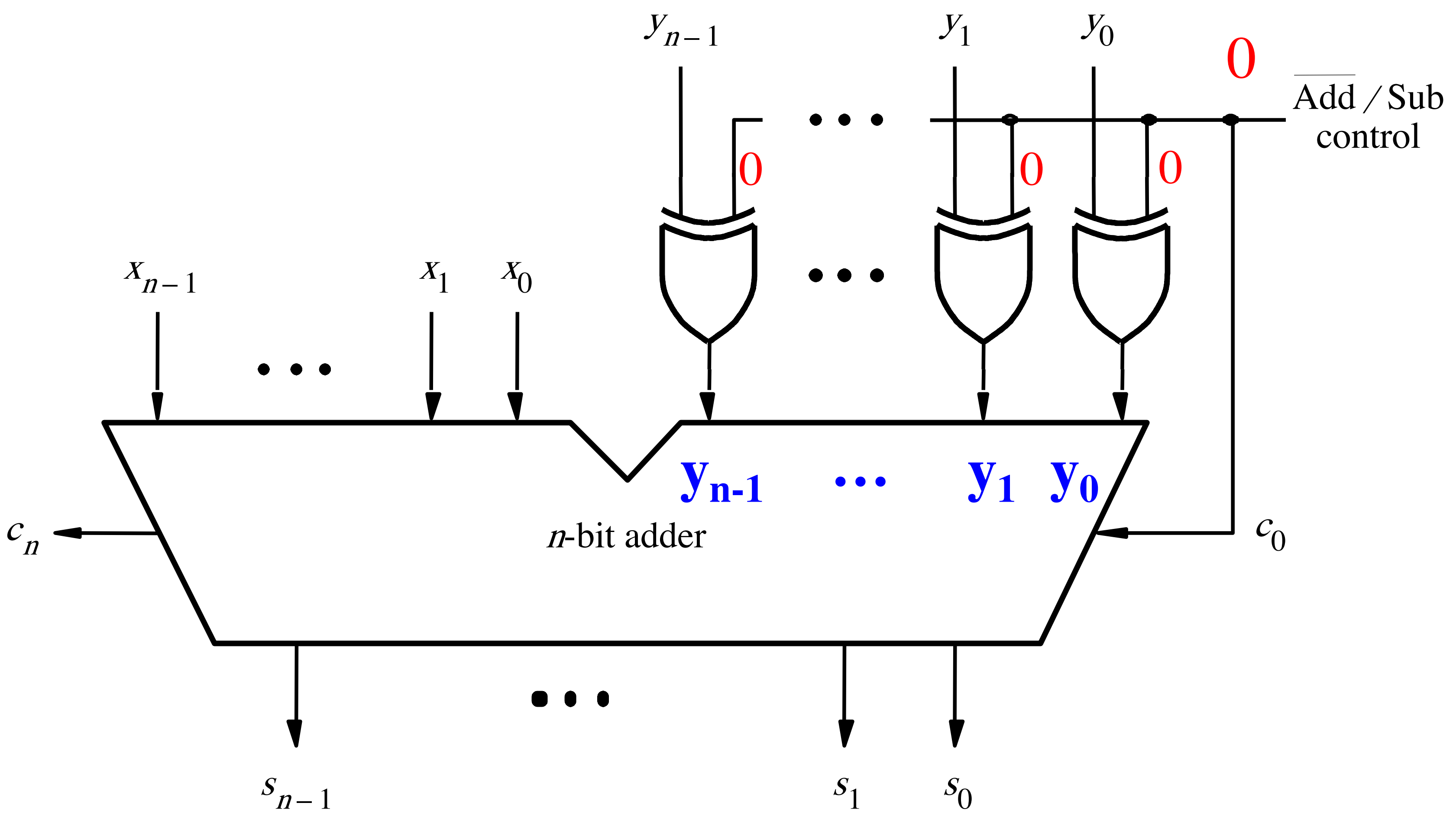
[Figure 3.12 from the textbook]

Addition: when control = 0



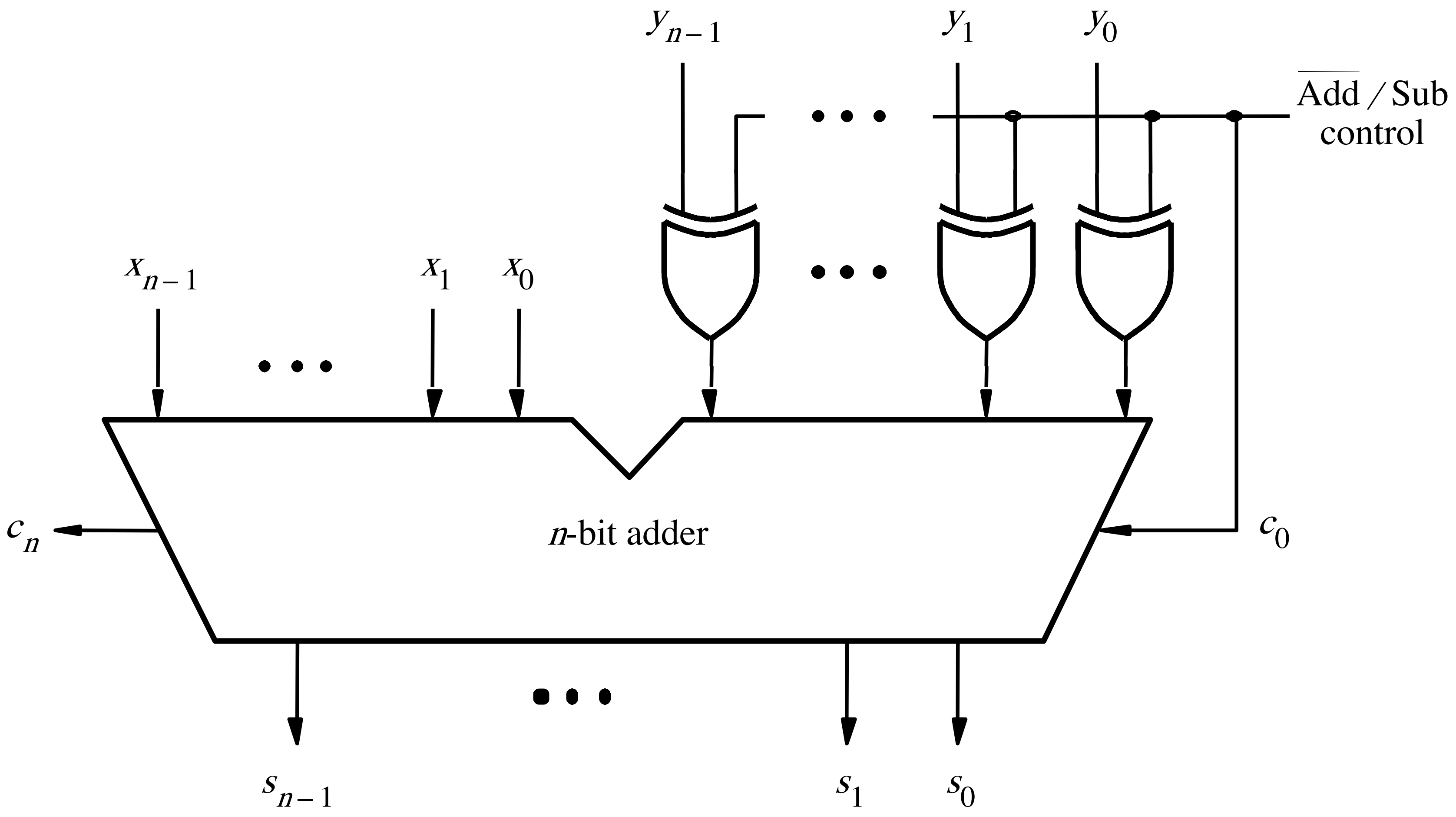
[Figure 3.12 from the textbook]

Addition: when control = 0



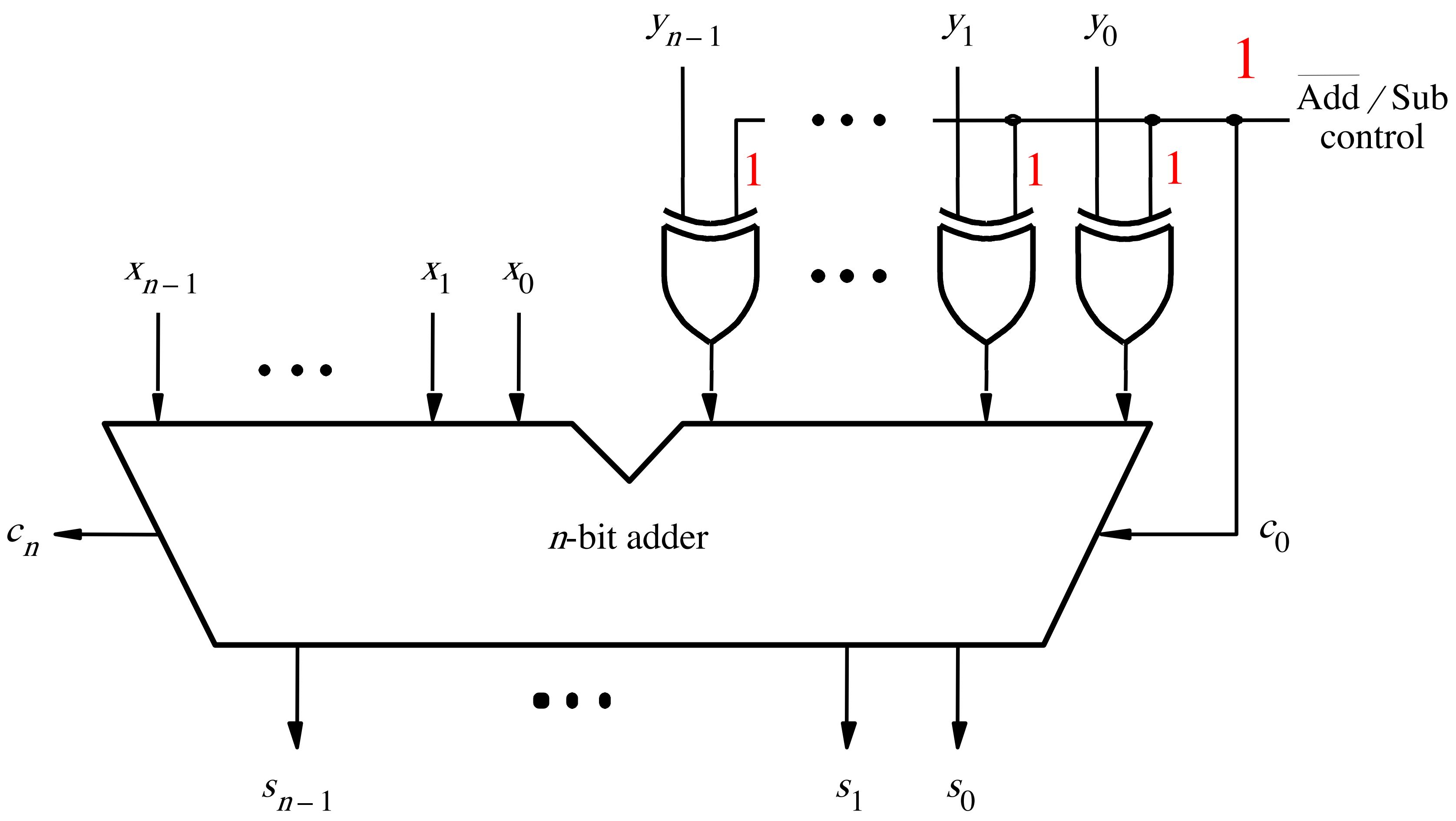
[Figure 3.12 from the textbook]

Subtraction: when control = 1



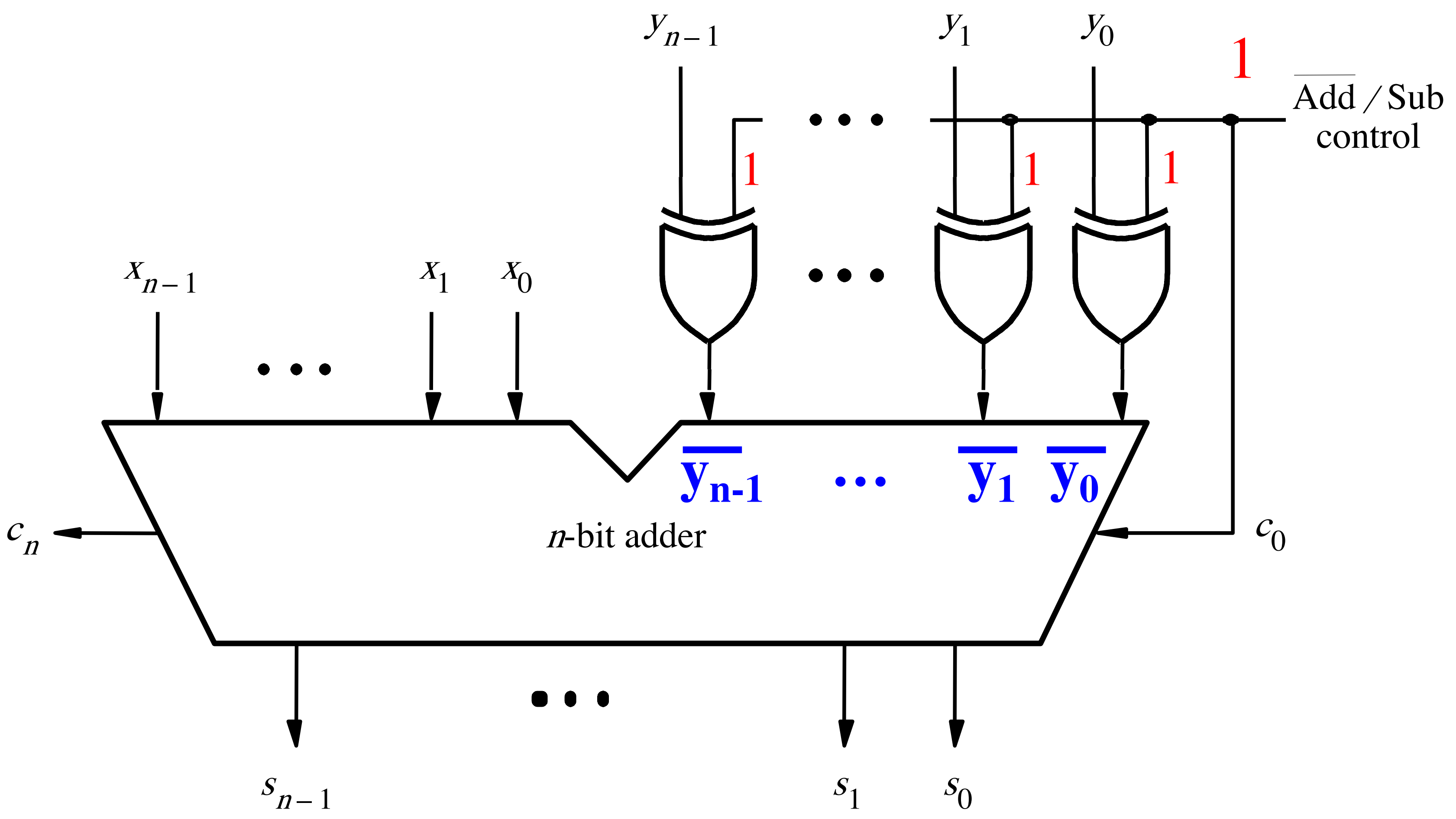
[Figure 3.12 from the textbook]

Subtraction: when control = 1



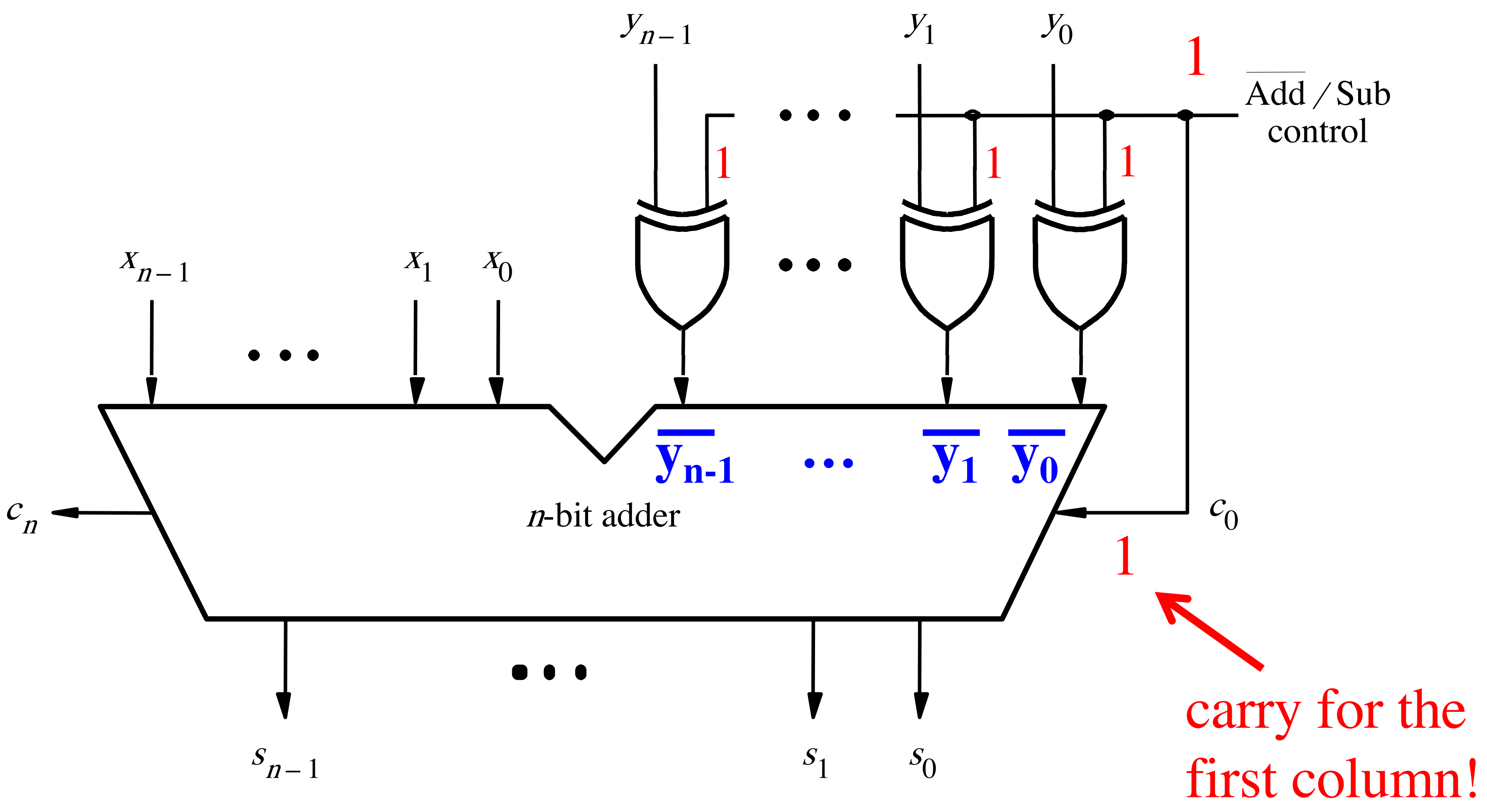
[Figure 3.12 from the textbook]

Subtraction: when control = 1



[Figure 3.12 from the textbook]

Subtraction: when control = 1



[Figure 3.12 from the textbook]

Overflow Detection

Examples of determination of overflow

$$\begin{array}{rr} & (+7) \\ + & (+2) \\ \hline & (+9) \end{array} \qquad \begin{array}{rr} & 0\,1\,1\,1 \\ + & 0\,0\,1\,0 \\ \hline & 1\,0\,0\,1 \end{array} \qquad\qquad \begin{array}{rr} & (-7) \\ + & (+2) \\ \hline & (-5) \end{array} \qquad \begin{array}{rr} & 1\,0\,0\,1 \\ + & 0\,0\,1\,0 \\ \hline & 1\,0\,1\,1 \end{array}$$

$$\begin{array}{rr} & (+7) \\ + & (-2) \\ \hline & (+5) \end{array} \qquad \begin{array}{rr} & 0\,1\,1\,1 \\ + & 1\,1\,1\,0 \\ \hline & 1\,0\,1\,0\,1 \end{array} \qquad\qquad \begin{array}{rr} & (-7) \\ + & (-2) \\ \hline & (-9) \end{array} \qquad \begin{array}{rr} & 1\,0\,0\,1 \\ + & 1\,1\,1\,0 \\ \hline & 1\,0\,1\,1\,1 \end{array}$$

[Figure 3.13 from the textbook]

Examples of determination of overflow

$$\begin{array}{rr} & (+7) \\ + & (+2) \\ \hline & \boxed{(+9)} \end{array} \qquad \begin{array}{rr} & 0\,1\,1\,1 \\ + & 0\,0\,1\,0 \\ \hline & 1\,0\,0\,1 \end{array} \qquad\qquad \begin{array}{rr} & (-7) \\ + & (+2) \\ \hline & (-5) \end{array} \qquad \begin{array}{rr} & 1\,0\,0\,1 \\ + & 0\,0\,1\,0 \\ \hline & 1\,0\,1\,1 \end{array}$$

$$\begin{array}{rr} & (+7) \\ + & (-2) \\ \hline & (+5) \end{array} \qquad \begin{array}{rr} & 0\,1\,1\,1 \\ + & 1\,1\,1\,0 \\ \hline & 1\,0\,1\,0\,1 \end{array} \qquad\qquad \begin{array}{rr} & (-7) \\ + & (-2) \\ \hline & \boxed{(-9)} \end{array} \qquad \begin{array}{rr} & 1\,0\,0\,1 \\ + & 1\,1\,1\,0 \\ \hline & 1\,0\,1\,1\,1 \end{array}$$

In 2's complement, both +9 and -9 are not representable with 4 bits.

[Figure 3.13 from the textbook]

Examples of determination of overflow

$$\begin{array}{rr}
 & & 0\,1\,1\,0\,0 \\
 (+7) & & 0\,1\,1\,1 \\
 +\ (+2) & + & 0\,0\,1\,0 \\
\hline
 (+9) & & 1\,0\,0\,1
\end{array}
\qquad\qquad
\begin{array}{rrr}
 & & 0\,0\,0\,0\,0 \\
 (-7) & & 1\,0\,0\,1 \\
 +\ (+2) & + & 0\,0\,1\,0 \\
\hline
 (-5) & & 1\,0\,1\,1
\end{array}$$

$$\begin{array}{rrr}
 & & 1\,1\,1\,0\,0 \\
 (+7) & & 0\,1\,1\,1 \\
 +\ (-2) & + & 1\,1\,1\,0 \\
\hline
 (+5) & & 1\;0\,1\,0\,1
\end{array}
\qquad\qquad
\begin{array}{rrr}
 & & 1\,0\,0\,0\,0 \\
 (-7) & & 1\,0\,0\,1 \\
 +\ (-2) & + & 1\,1\,1\,0 \\
\hline
 (-9) & & 1\,0\,1\,1\,1
\end{array}$$

Include the carry bits: $c_4\ c_3\ c_2\ c_1\ c_0$

Examples of determination of overflow

$$\begin{array}{rr} & (+7) \\ + & (+2) \\ \hline & (+9) \end{array} \qquad \begin{array}{rr} & \boxed{0\ 1}\,1\ 0\ 0 \\ & 0\ 1\ 1\ 1 \\ + & 0\ 0\ 1\ 0 \\ \hline & 1\ 0\ 0\ 1 \end{array} \qquad\qquad \begin{array}{rr} & (-7) \\ + & (+2) \\ \hline & (-5) \end{array} \qquad \begin{array}{rr} & \boxed{0\ 0}\,0\ 0\ 0 \\ & 1\ 0\ 0\ 1 \\ + & 0\ 0\ 1\ 0 \\ \hline & 1\ 0\ 1\ 1 \end{array}$$

$$\begin{array}{rr} & (+7) \\ + & (-2) \\ \hline & (+5) \end{array} \qquad \begin{array}{rr} & \boxed{1\ 1}\,1\ 0\ 0 \\ & 0\ 1\ 1\ 1 \\ + & 1\ 1\ 1\ 0 \\ \hline & 1\ 0\ 1\ 0\ 1 \end{array} \qquad\qquad \begin{array}{rr} & (-7) \\ + & (-2) \\ \hline & (-9) \end{array} \qquad \begin{array}{rr} & \boxed{1\ 0}\,0\ 0\ 0 \\ & 1\ 0\ 0\ 1 \\ + & 1\ 1\ 1\ 0 \\ \hline & 1\ 0\ 1\ 1\ 1 \end{array}$$

Include the carry bits: $\boxed{c_4\ c_3}\,c_2\ c_1\ c_0$

Examples of determination of overflow

$c_4 = 0$
$c_3 = 1$

$$\begin{array}{rr} & \boxed{0\;1}\,1\;0\;0 \\ (+7) & 0\;1\;1\;1 \\ +\;(+2) & +\quad 0\;0\;1\;0 \\ \hline (+9) & 1\;0\;0\;1 \end{array}$$

$c_4 = 0$
$c_3 = 0$

$$\begin{array}{rr} & \boxed{0\;0}\,0\;0\;0 \\ (-7) & 1\;0\;0\;1 \\ +\;(+2) & +\quad 0\;0\;1\;0 \\ \hline (-5) & 1\;0\;1\;1 \end{array}$$

$c_4 = 1$
$c_3 = 1$

$$\begin{array}{rr} & \boxed{1\;1}\,1\;0\;0 \\ (+7) & 0\;1\;1\;1 \\ +\;(-2) & +\quad 1\;1\;1\;0 \\ \hline (+5) & 1\;0\;1\;0\;1 \end{array}$$

$c_4 = 1$
$c_3 = 0$

$$\begin{array}{rr} & \boxed{1\;0}\,0\;0\;0 \\ (-7) & 1\;0\;0\;1 \\ +\;(-2) & +\quad 1\;1\;1\;0 \\ \hline (-9) & 1\;0\;1\;1\;1 \end{array}$$

Include the carry bits: $\boxed{c_4\ c_3}\,c_2\ c_1\ c_0$

Examples of determination of overflow

$c_4 = 0$
$c_3 = 1$

$$\begin{array}{rr} & (+7) \\ + & (+2) \\ \hline & (+9) \end{array} \qquad \begin{array}{rr} & \boxed{0\;1}\,1\;0\;0 \\ & 0\;1\;1\;1 \\ + & 0\;0\;1\;0 \\ \hline & 1\;0\;0\;1 \end{array}$$

$$\begin{array}{rr} & (-7) \\ + & (+2) \\ \hline & (-5) \end{array} \qquad \begin{array}{rr} & \boxed{0\;0}\,0\;0\;0 \\ & 1\;0\;0\;1 \\ + & 0\;0\;1\;0 \\ \hline & 1\;0\;1\;1 \end{array}$$

$c_4 = 0$
$c_3 = 0$

$c_4 = 1$
$c_3 = 1$

$$\begin{array}{rr} & (+7) \\ + & (-2) \\ \hline & (+5) \end{array} \qquad \begin{array}{rr} & \boxed{1\;1}\,1\;0\;0 \\ & 0\;1\;1\;1 \\ + & 1\;1\;1\;0 \\ \hline & 1\;0\;1\;0\;1 \end{array}$$

$$\begin{array}{rr} & (-7) \\ + & (-2) \\ \hline & (-9) \end{array} \qquad \begin{array}{rr} & \boxed{1\;0}\,0\;0\;0 \\ & 1\;0\;0\;1 \\ + & 1\;1\;1\;0 \\ \hline & 1\;0\;1\;1\;1 \end{array}$$

$c_4 = 1$
$c_3 = 0$

Overflow occurs only in these two cases.

Examples of determination of overflow

$c_4 = 0$
$c_3 = 1$

$$\begin{array}{r} (+7) \\ +\ (+2) \\ \hline (+9) \end{array} \qquad \begin{array}{rr} & \boxed{0\ 1}1\ 0\ 0 \\ & 0\ 1\ 1\ 1 \\ + & 0\ 0\ 1\ 0 \\ \hline & 1\ 0\ 0\ 1 \end{array}$$

$$\begin{array}{r} (-7) \\ +\ (+2) \\ \hline (-5) \end{array} \qquad \begin{array}{rr} & \boxed{0\ 0}0\ 0\ 0 \\ & 1\ 0\ 0\ 1 \\ + & 0\ 0\ 1\ 0 \\ \hline & 1\ 0\ 1\ 1 \end{array}$$

$c_4 = 0$
$c_3 = 0$

$c_4 = 1$
$c_3 = 1$

$$\begin{array}{r} (+7) \\ +\ (-2) \\ \hline (+5) \end{array} \qquad \begin{array}{rr} & \boxed{1\ 1}1\ 0\ 0 \\ & 0\ 1\ 1\ 1 \\ + & 1\ 1\ 1\ 0 \\ \hline & 1\ 0\ 1\ 0\ 1 \end{array}$$

$$\begin{array}{r} (-7) \\ +\ (-2) \\ \hline (-9) \end{array} \qquad \begin{array}{rr} & \boxed{1\ 0}0\ 0\ 0 \\ & 1\ 0\ 0\ 1 \\ + & 1\ 1\ 1\ 0 \\ \hline & 1\ 0\ 1\ 1\ 1 \end{array}$$

$c_4 = 1$
$c_3 = 0$

$$\text{Overflow} = c_3\overline{c}_4 + \overline{c}_3 c_4$$

Examples of determination of overflow

$$c_4 = 0 \qquad c_3 = 1$$

$$\begin{array}{rr}
 & \boxed{0\ 1}\,1\ 0\ 0 \\
(+7) & 0\ 1\ 1\ 1 \\
+\ (+2) & +\quad 0\ 0\ 1\ 0 \\
\hline
(+9) & 1\ 0\ 0\ 1
\end{array}$$

$$\begin{array}{rr}
 & \boxed{0\ 0}\,0\ 0\ 0 \\
(-7) & 1\ 0\ 0\ 1 \\
+\ (+2) & +\quad 0\ 0\ 1\ 0 \\
\hline
(-5) & 1\ 0\ 1\ 1
\end{array}$$

$$c_4 = 0 \qquad c_3 = 0$$

$$c_4 = 1 \qquad c_3 = 1$$

$$\begin{array}{rr}
 & \boxed{1\ 1}\,1\ 0\ 0 \\
(+7) & 0\ 1\ 1\ 1 \\
+\ (-2) & +\quad 1\ 1\ 1\ 0 \\
\hline
(+5) & 1\ 0\ 1\ 0\ 1
\end{array}$$

$$\begin{array}{rr}
 & \boxed{1\ 0}\,0\ 0\ 0 \\
(-7) & 1\ 0\ 0\ 1 \\
+\ (-2) & +\quad 1\ 1\ 1\ 0 \\
\hline
(-9) & 1\ 0\ 1\ 1\ 1
\end{array}$$

$$c_4 = 1 \qquad c_3 = 0$$

$$\text{Overflow} = \underbrace{c_3\overline{c}_4 + \overline{c}_3 c_4}_{\text{XOR}}$$

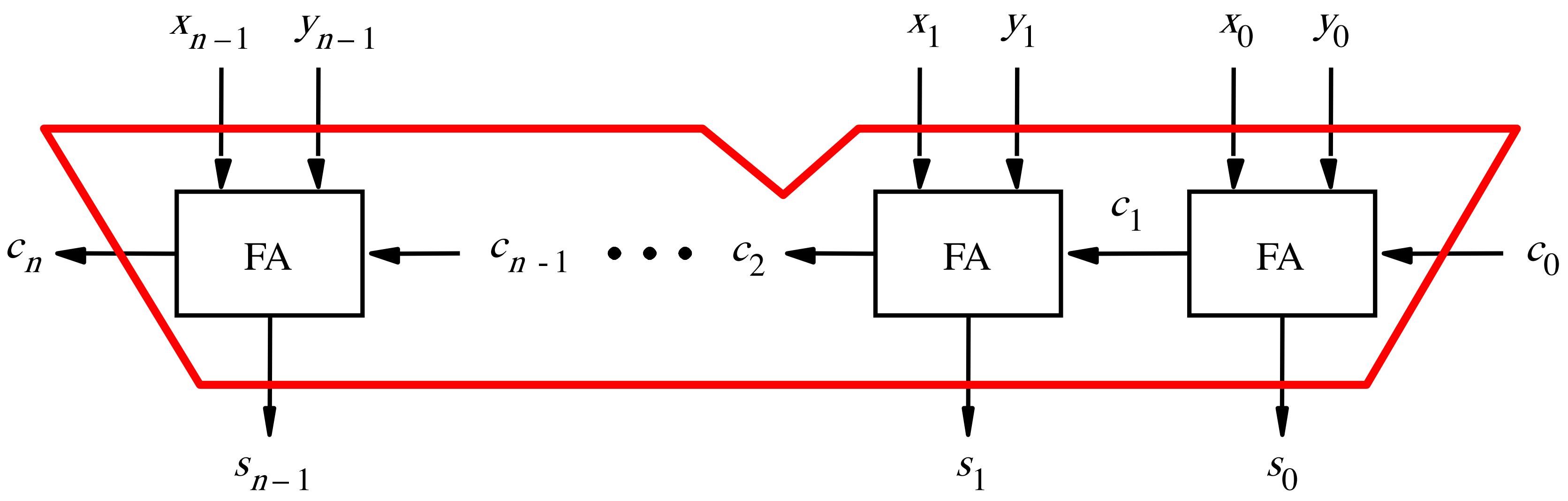
Calculating overflow for 4-bit numbers with only three significant bits

$$\begin{aligned}\text{Overflow} &= c_3\overline{c}_4 + \overline{c}_3 c_4 \\ &= c_3 \oplus c_4\end{aligned}$$

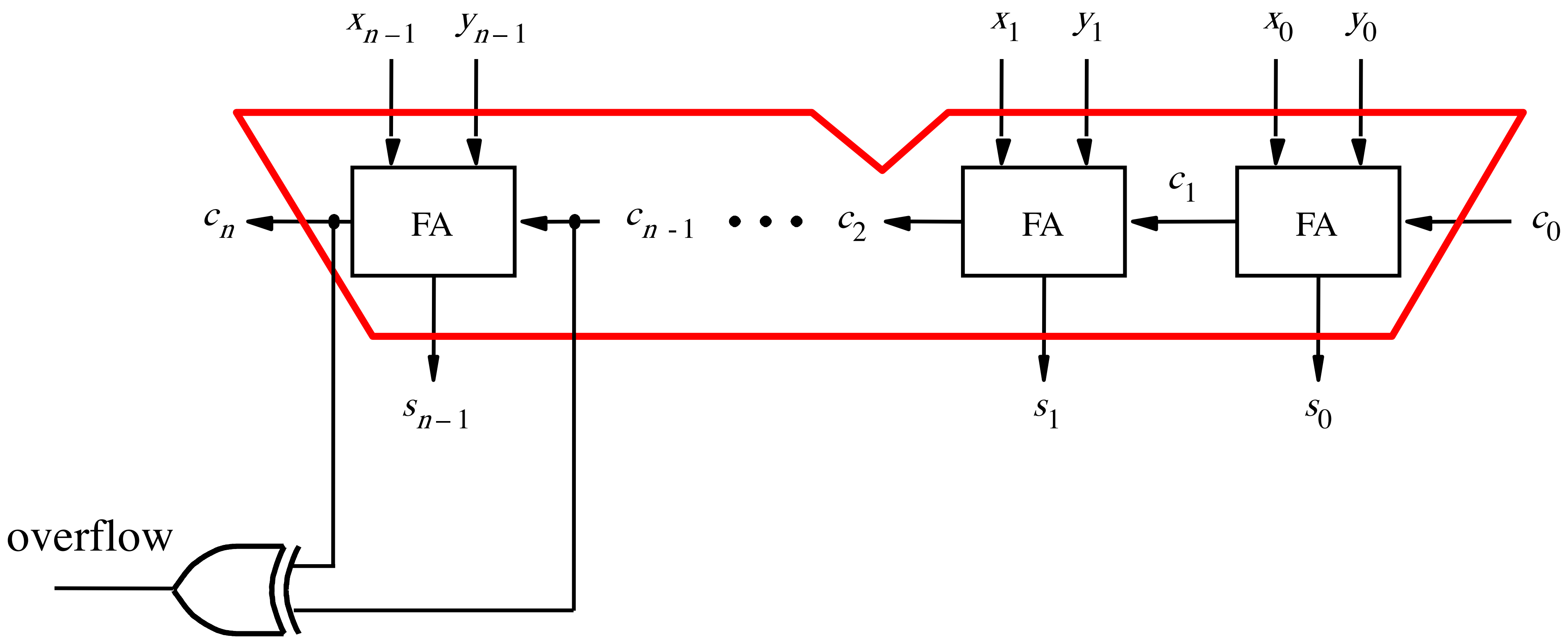
Calculating overflow for n-bit numbers with only n-1 significant bits

$$\text{Overflow} = c_{n-1} \oplus c_n$$

Detecting Overflow



Detecting Overflow (with one extra XOR)



Overflow Detection (alternative method)

Another way to look at the overflow issue

$$\begin{array}{rl} & X = x_3\ x_2\ x_1\ x_0 \\ + & Y = y_3\ y_2\ y_1\ y_0 \\ \hline & S = s_3\ s_2\ s_1\ s_0 \end{array}$$

Another way to look at the overflow issue

$$\begin{array}{rr|c|ccc} & X= & x_3 & x_2 & x_1 & x_0 \\ + & Y= & y_3 & y_2 & y_1 & y_0 \\ \hline & S= & s_3 & s_2 & s_1 & s_0 \end{array}$$

If both numbers that we are adding have the same sign but the sum does not, then we have an overflow.

Examples of determination of overflow

$$\begin{array}{r} (+7) \\ +\ (+2) \\ \hline (+9) \end{array} \qquad \begin{array}{rr} & 0\,1\,1\,1 \\ + & 0\,0\,1\,0 \\ \hline & 1\,0\,0\,1 \end{array} \qquad\qquad \begin{array}{r} (-7) \\ +\ (+2) \\ \hline (-5) \end{array} \qquad \begin{array}{rr} & 1\,0\,0\,1 \\ + & 0\,0\,1\,0 \\ \hline & 1\,0\,1\,1 \end{array}$$

$$\begin{array}{r} (+7) \\ +\ (-2) \\ \hline (+5) \end{array} \qquad \begin{array}{rr} & 0\,1\,1\,1 \\ + & 1\,1\,1\,0 \\ \hline & 1\,0\,1\,0\,1 \end{array} \qquad\qquad \begin{array}{r} (-7) \\ +\ (-2) \\ \hline (-9) \end{array} \qquad \begin{array}{rr} & 1\,0\,0\,1 \\ + & 1\,1\,1\,0 \\ \hline & 1\,0\,1\,1\,1 \end{array}$$

Examples of determination of overflow

$$\begin{array}{r}(+7)\\+\ (+2)\\\hline(+9)\end{array}\qquad\begin{array}{cc|c|ccc} & & 0&1&1&1\\ + & & 0&0&1&0\\\hline & & 1&0&0&1\end{array}\qquad\qquad\begin{array}{r}(-7)\\+\ (+2)\\\hline(-5)\end{array}\qquad\begin{array}{cc|c|ccc} & & 1&0&0&1\\ + & & 0&0&1&0\\\hline & & 1&0&1&1\end{array}$$

$$\begin{array}{r}(+7)\\+\ (-2)\\\hline(+5)\end{array}\qquad\begin{array}{cc|c|ccc} & & 0&1&1&1\\ + & & 1&1&1&0\\\hline & 1 & 0&1&0&1\end{array}\qquad\qquad\begin{array}{r}(-7)\\+\ (-2)\\\hline(-9)\end{array}\qquad\begin{array}{cc|c|ccc} & & 1&0&0&1\\ + & & 1&1&1&0\\\hline & 1 & 0&1&1&1\end{array}$$

Examples of determination of overflow

$x_3 = 0$
$y_3 = 0$
$s_3 = 1$

$$\begin{array}{r} (+7) \\ +\ (+2) \\ \hline (+9) \end{array} \qquad \begin{array}{cc|c|ccc} & + & 0 & 1 & 1 & 1 \\ & & 0 & 0 & 1 & 0 \\ \hline & & 1 & 0 & 0 & 1 \end{array}$$

$x_3 = 1$
$y_3 = 0$
$s_3 = 1$

$$\begin{array}{r} (-7) \\ +\ (+2) \\ \hline (-5) \end{array} \qquad \begin{array}{cc|c|ccc} & + & 1 & 0 & 0 & 1 \\ & & 0 & 0 & 1 & 0 \\ \hline & & 1 & 0 & 1 & 1 \end{array}$$

$x_3 = 0$
$y_3 = 1$
$s_3 = 0$

$$\begin{array}{r} (+7) \\ +\ (-2) \\ \hline (+5) \end{array} \qquad \begin{array}{cc|c|ccc} & + & 0 & 1 & 1 & 1 \\ & & 1 & 1 & 1 & 0 \\ \hline & 1 & 0 & 1 & 0 & 1 \end{array}$$

$x_3 = 1$
$y_3 = 1$
$s_3 = 0$

$$\begin{array}{r} (-7) \\ +\ (-2) \\ \hline (-9) \end{array} \qquad \begin{array}{cc|c|ccc} & + & 1 & 0 & 0 & 1 \\ & & 1 & 1 & 1 & 0 \\ \hline & 1 & 0 & 1 & 1 & 1 \end{array}$$

Examples of determination of overflow

$$\begin{array}{l} x_3 = 0 \\ y_3 = 0 \\ s_3 = 1 \end{array} \qquad \begin{array}{rr} & (+7) \\ + & (+2) \\ \hline & \boxed{(+9)} \end{array} \qquad + \begin{array}{r} 0\,1\,1\,1 \\ 0\,0\,1\,0 \\ \hline 1\,0\,0\,1 \end{array}$$

$$\begin{array}{rr} & (-7) \\ + & (+2) \\ \hline & (-5) \end{array} \qquad + \begin{array}{r} 1\,0\,0\,1 \\ 0\,0\,1\,0 \\ \hline 1\,0\,1\,1 \end{array} \qquad \begin{array}{l} x_3 = 1 \\ y_3 = 0 \\ s_3 = 1 \end{array}$$

$$\begin{array}{l} x_3 = 0 \\ y_3 = 1 \\ s_3 = 0 \end{array} \qquad \begin{array}{rr} & (+7) \\ + & (-2) \\ \hline & (+5) \end{array} \qquad + \begin{array}{r} 0\,1\,1\,1 \\ 1\,1\,1\,0 \\ \hline 1\,0\,1\,0\,1 \end{array}$$

$$\begin{array}{rr} & (-7) \\ + & (-2) \\ \hline & \boxed{(-9)} \end{array} \qquad + \begin{array}{r} 1\,0\,0\,1 \\ 1\,1\,1\,0 \\ \hline 1\,0\,1\,1\,1 \end{array} \qquad \begin{array}{l} x_3 = 1 \\ y_3 = 1 \\ s_3 = 0 \end{array}$$

In 2's complement, both +9 and -9 are not representable with 4 bits.

Examples of determination of overflow

$$\begin{aligned} x_3 &= 0 \\ y_3 &= 0 \\ s_3 &= 1 \end{aligned}$$

$$\begin{array}{r} (+7) \\ +\ (+2) \\ \hline (+9) \end{array} \qquad \begin{array}{rr} & 0\,1\,1\,1 \\ + & 0\,0\,1\,0 \\ \hline & 1\,0\,0\,1 \end{array}$$

$$\begin{array}{r} (-7) \\ +\ (+2) \\ \hline (-5) \end{array} \qquad \begin{array}{rr} & 1\,0\,0\,1 \\ + & 0\,0\,1\,0 \\ \hline & 1\,0\,1\,1 \end{array}$$

$$\begin{aligned} x_3 &= 1 \\ y_3 &= 0 \\ s_3 &= 1 \end{aligned}$$

$$\begin{aligned} x_3 &= 0 \\ y_3 &= 1 \\ s_3 &= 0 \end{aligned}$$

$$\begin{array}{r} (+7) \\ +\ (-2) \\ \hline (+5) \end{array} \qquad \begin{array}{rr} & 0\,1\,1\,1 \\ + & 1\,1\,1\,0 \\ \hline & 1\,0\,1\,0\,1 \end{array}$$

$$\begin{array}{r} (-7) \\ +\ (-2) \\ \hline (-9) \end{array} \qquad \begin{array}{rr} & 1\,0\,0\,1 \\ + & 1\,1\,1\,0 \\ \hline & 1\,0\,1\,1\,1 \end{array}$$

$$\begin{aligned} x_3 &= 1 \\ y_3 &= 1 \\ s_3 &= 0 \end{aligned}$$

Overflow occurs only in these two cases.

Examples of determination of overflow

$x_3 = 0$
$y_3 = 0$
$s_3 = 1$

$$\begin{array}{r} (+7) \\ +\ (+2) \\ \hline (+9) \end{array} \qquad \begin{array}{r} 0\,1\,1\,1 \\ +\ 0\,0\,1\,0 \\ \hline 1\,0\,0\,1 \end{array}$$

$$\begin{array}{r} (-7) \\ +\ (+2) \\ \hline (-5) \end{array} \qquad \begin{array}{r} 1\,0\,0\,1 \\ +\ 0\,0\,1\,0 \\ \hline 1\,0\,1\,1 \end{array}$$

$x_3 = 1$
$y_3 = 0$
$s_3 = 1$

$x_3 = 0$
$y_3 = 1$
$s_3 = 0$

$$\begin{array}{r} (+7) \\ +\ (-2) \\ \hline (+5) \end{array} \qquad \begin{array}{r} 0\,1\,1\,1 \\ +\ 1\,1\,1\,0 \\ \hline 1\,0\,1\,0\,1 \end{array}$$

$$\begin{array}{r} (-7) \\ +\ (-2) \\ \hline (-9) \end{array} \qquad \begin{array}{r} 1\,0\,0\,1 \\ +\ 1\,1\,1\,0 \\ \hline 1\,0\,1\,1\,1 \end{array}$$

$x_3 = 1$
$y_3 = 1$
$s_3 = 0$

$$\text{Overflow} = \overline{x}_3\,\overline{y}_3\,s_3 + x_3\,y_3\,\overline{s}_3$$

Another way to look at the overflow issue

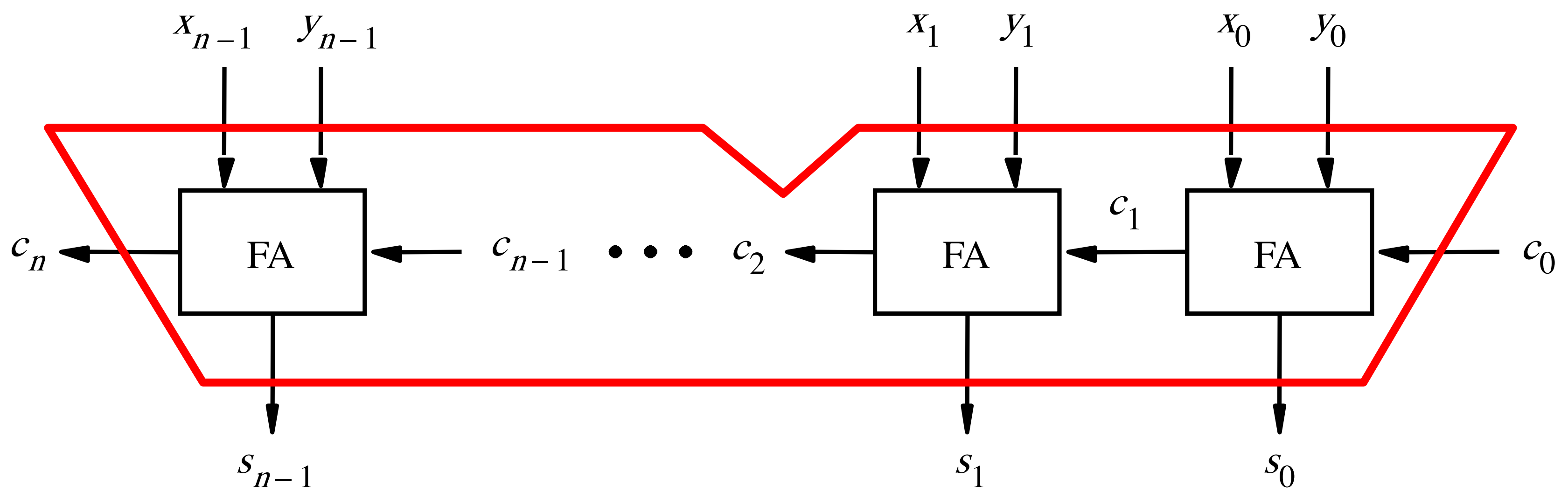
$$
\begin{array}{cr|c|ccc}
 & X= & x_3 & x_2 & x_1 & x_0 \\
+ & Y= & y_3 & y_2 & y_1 & y_0 \\
\hline
 & S= & s_3 & s_2 & s_1 & s_0
\end{array}
$$

If both numbers that we are adding have the same sign but the sum does not, then we have an overflow.

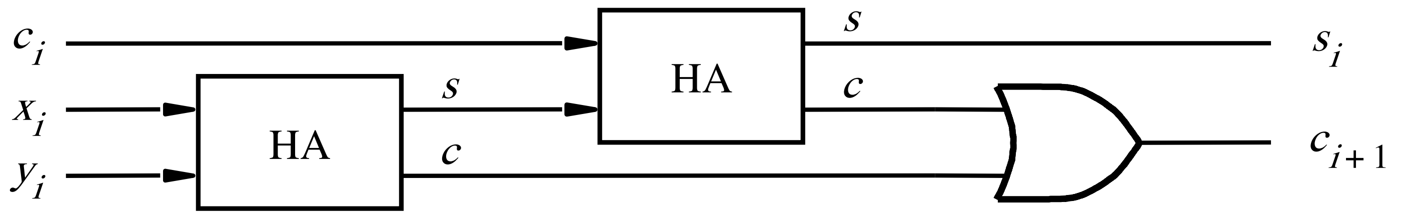
$$\text{Overflow} = \overline{x}_3\,\overline{y}_3\,s_3 + x_3\,y_3\,\overline{s}_3$$

A ripple-carry adder

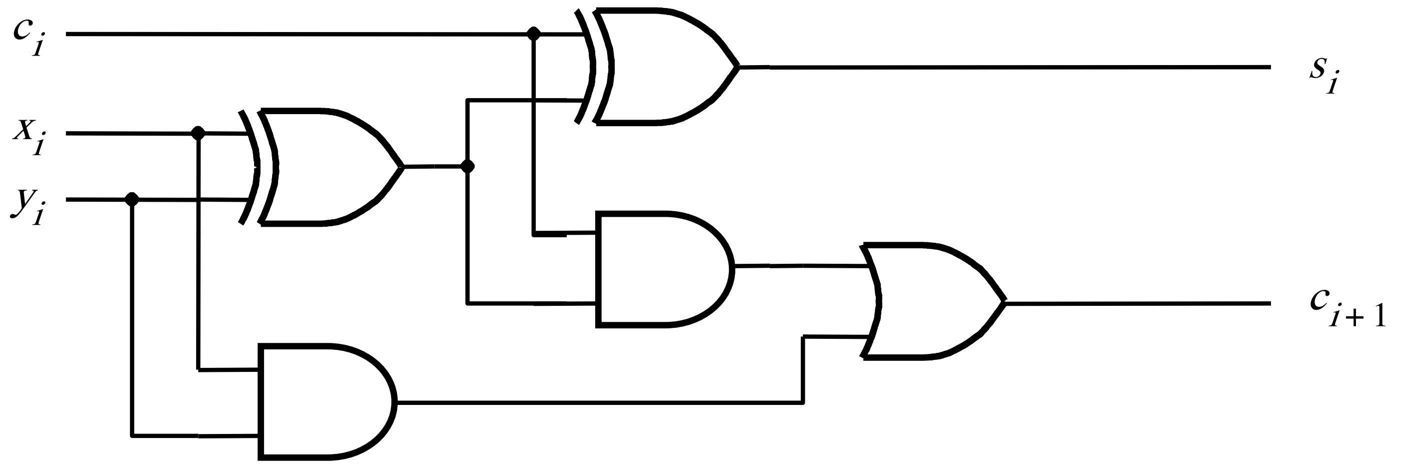
How long does it take to compute all sum bits and all carry bits?



Delays through the modular implementation of the full-adder circuit



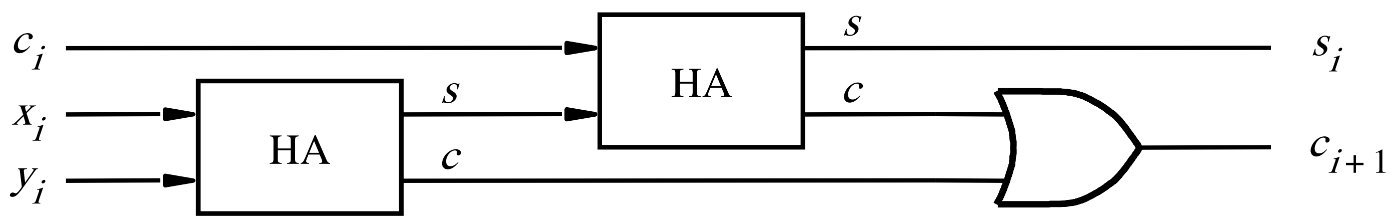
(a) Block diagram



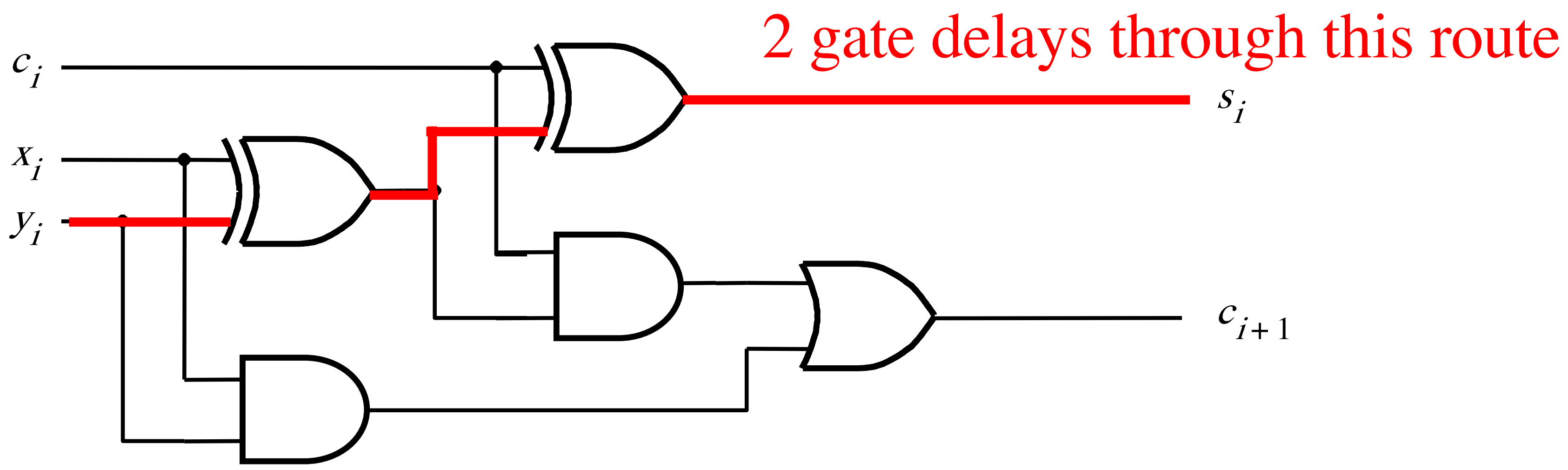
(b) Detailed diagram

[Figure 3.4 from the textbook]

Delays through the modular implementation of the full-adder circuit



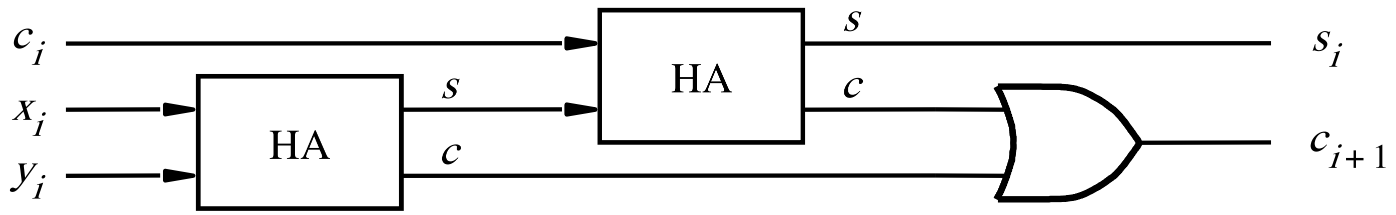
(a) Block diagram



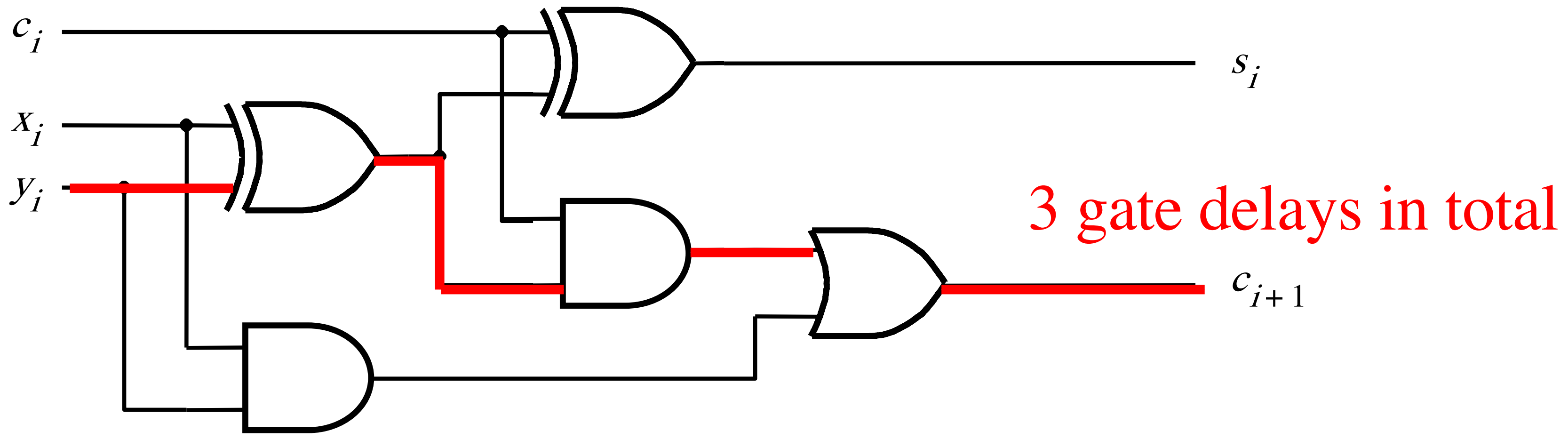
(b) Detailed diagram

[Figure 3.4 from the textbook]

Delays through the modular implementation of the full-adder circuit



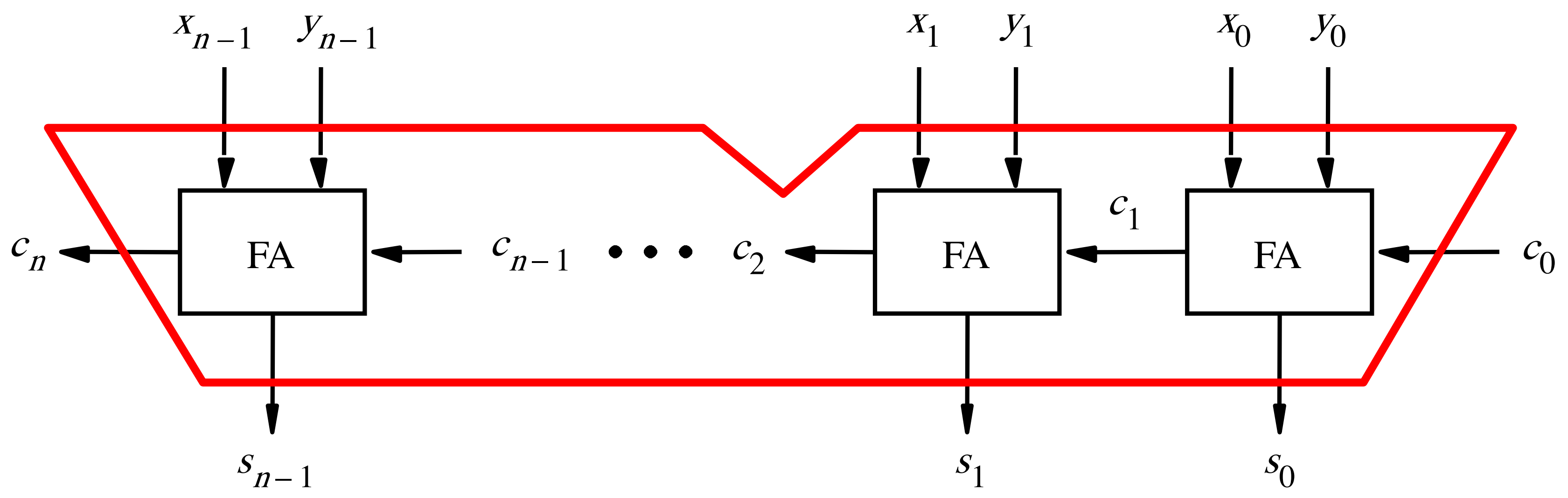
(a) Block diagram



(b) Detailed diagram

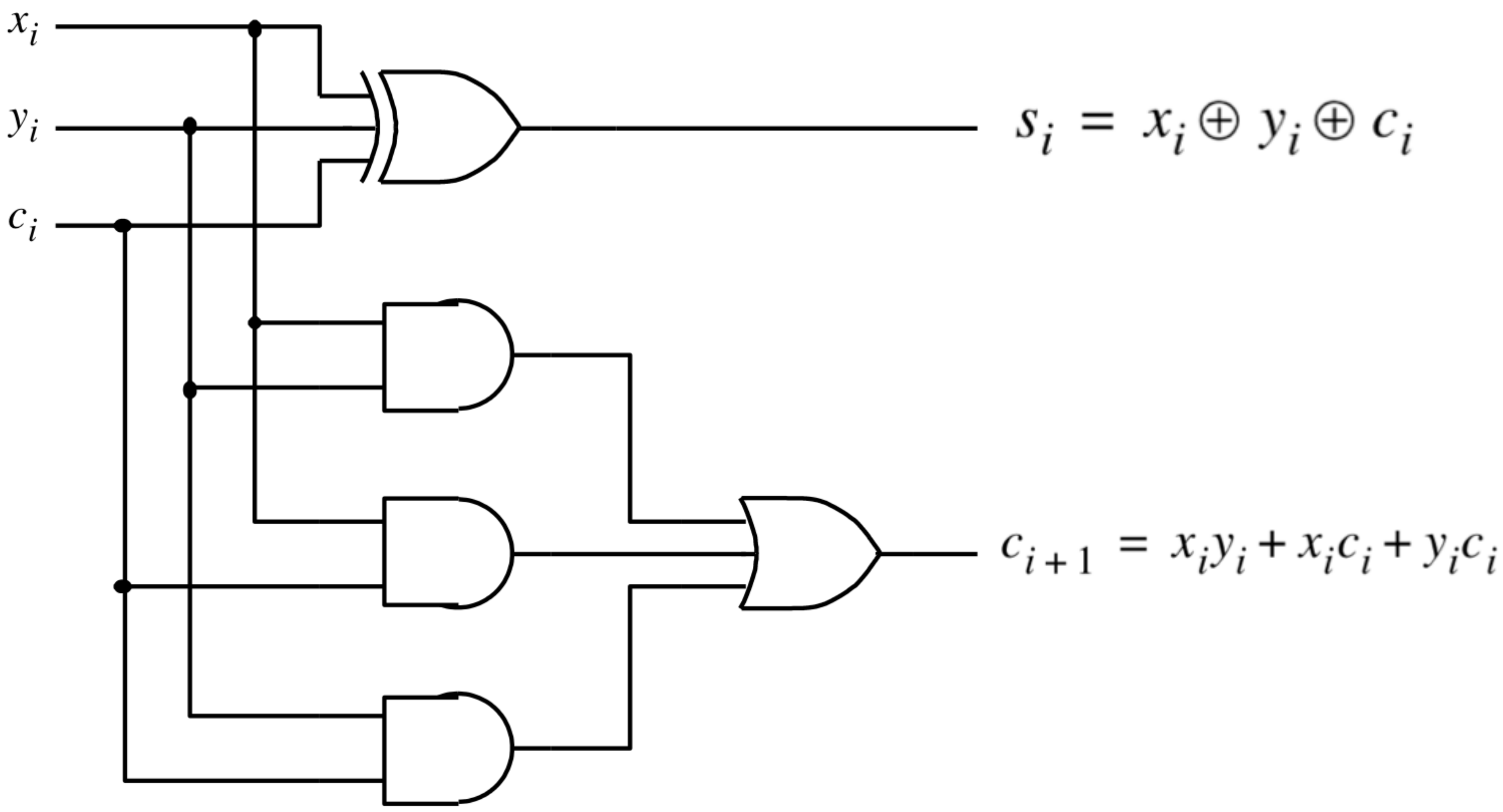
[Figure 3.4 from the textbook]

How long does it take to compute all sum bits and all carry bits in this case?



It takes 3n gate delays?

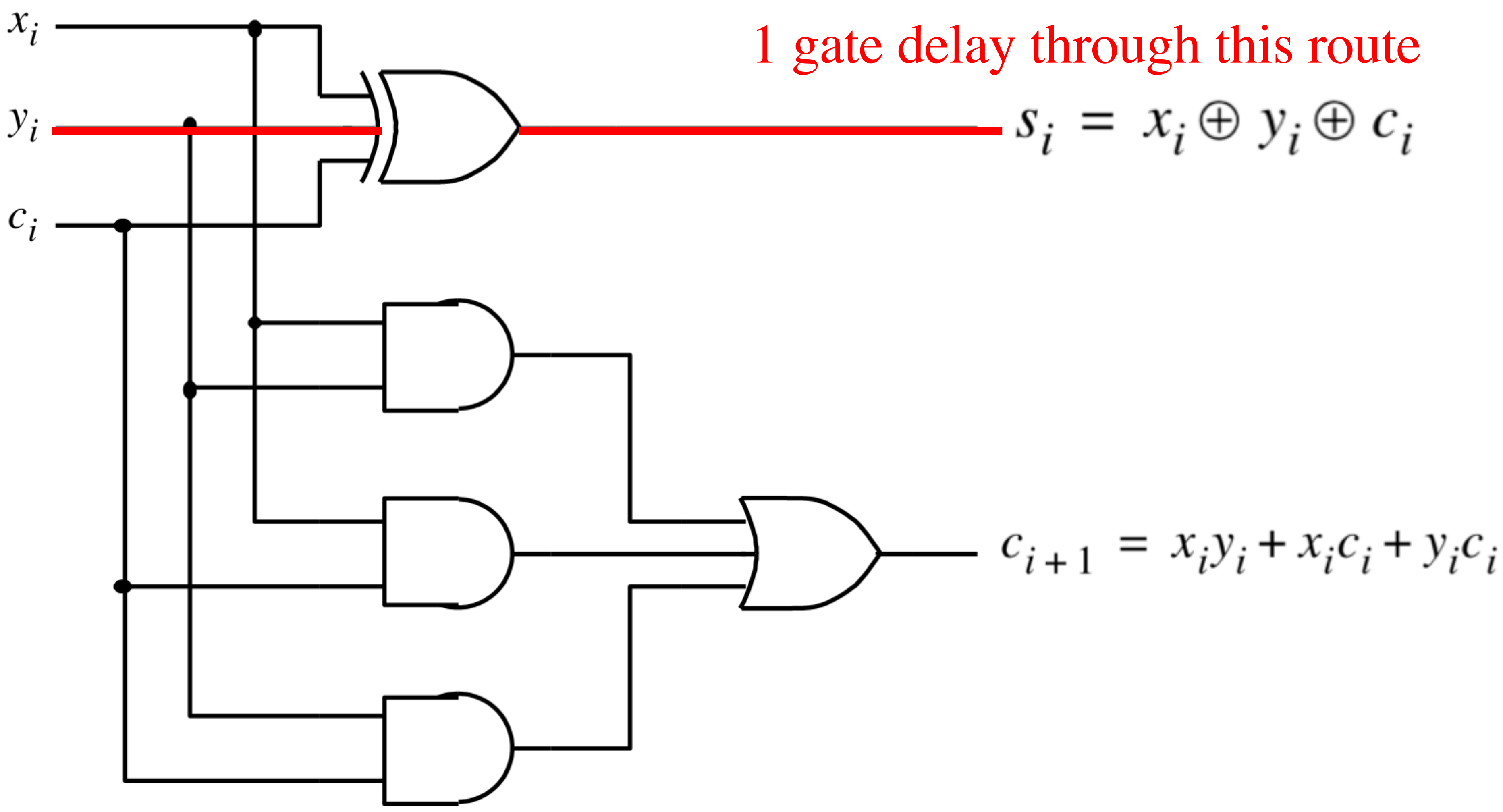
Delays through the Full-Adder circuit



x_i
y_i
c_i
$s_i = x_i \oplus y_i \oplus c_i$
$c_{i+1} = x_i y_i + x_i c_i + y_i c_i$

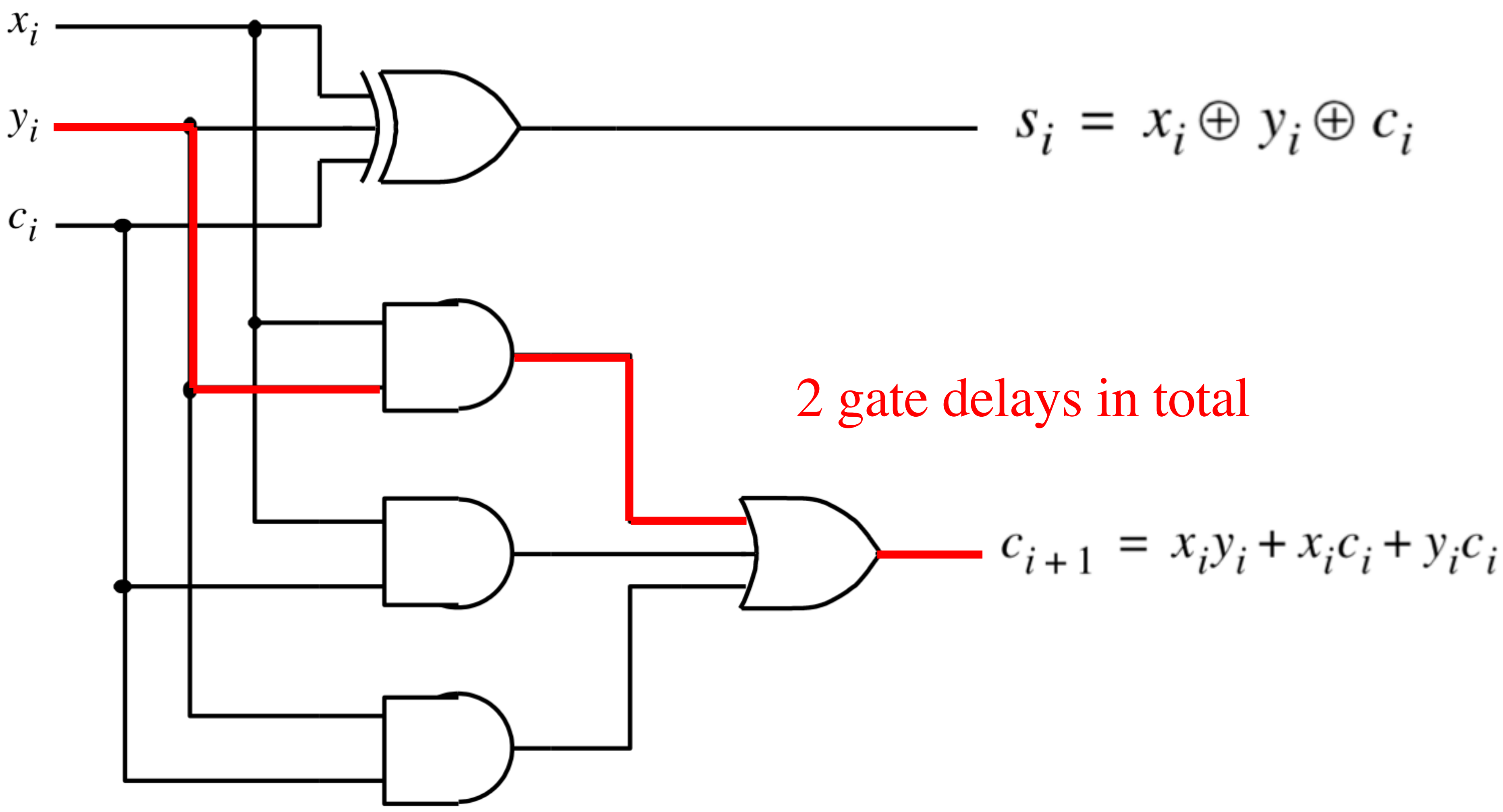
[Figure 3.3c from the textbook]

Delays through the Full-Adder circuit


x_i
y_i
c_i
1 gate delay through this route
$s_i = x_i \oplus y_i \oplus c_i$
$c_{i+1} = x_i y_i + x_i c_i + y_i c_i$

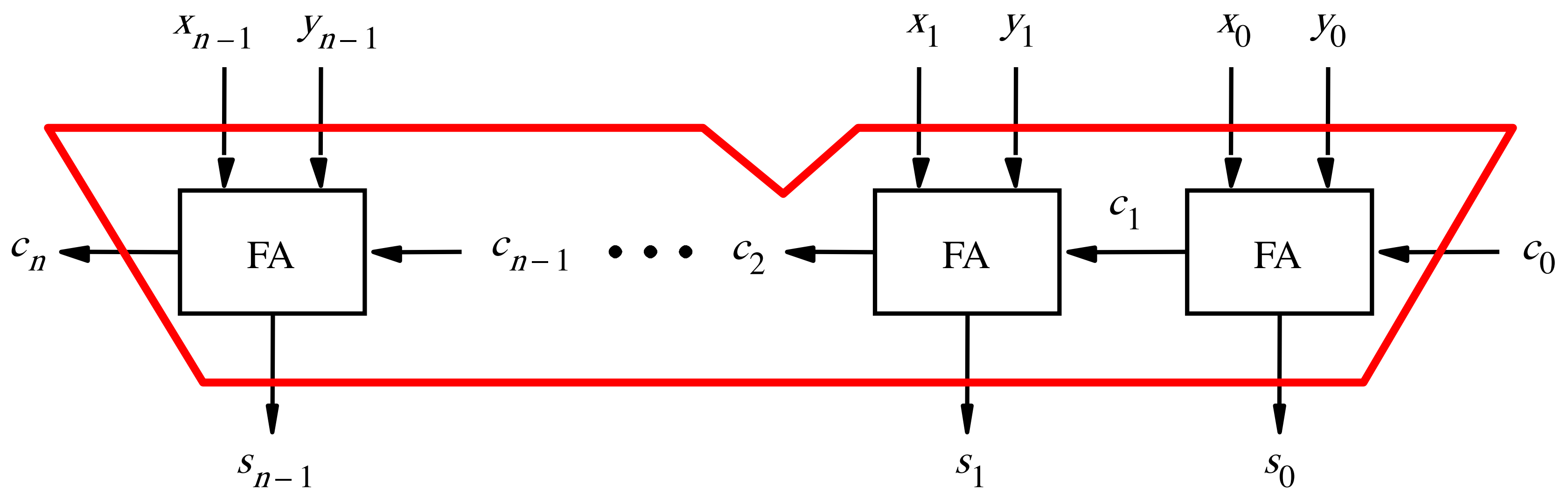
[Figure 3.3c from the textbook]

Delays through the Full-Adder circuit


x_i
y_i
c_i
$s_i = x_i \oplus y_i \oplus c_i$
2 gate delays in total
$c_{i+1} = x_i y_i + x_i c_i + y_i c_i$

[Figure 3.3c from the textbook]

How long does it take to compute all sum bits and all carry bits?



It takes 2n gate delays?

Can we perform addition even faster?

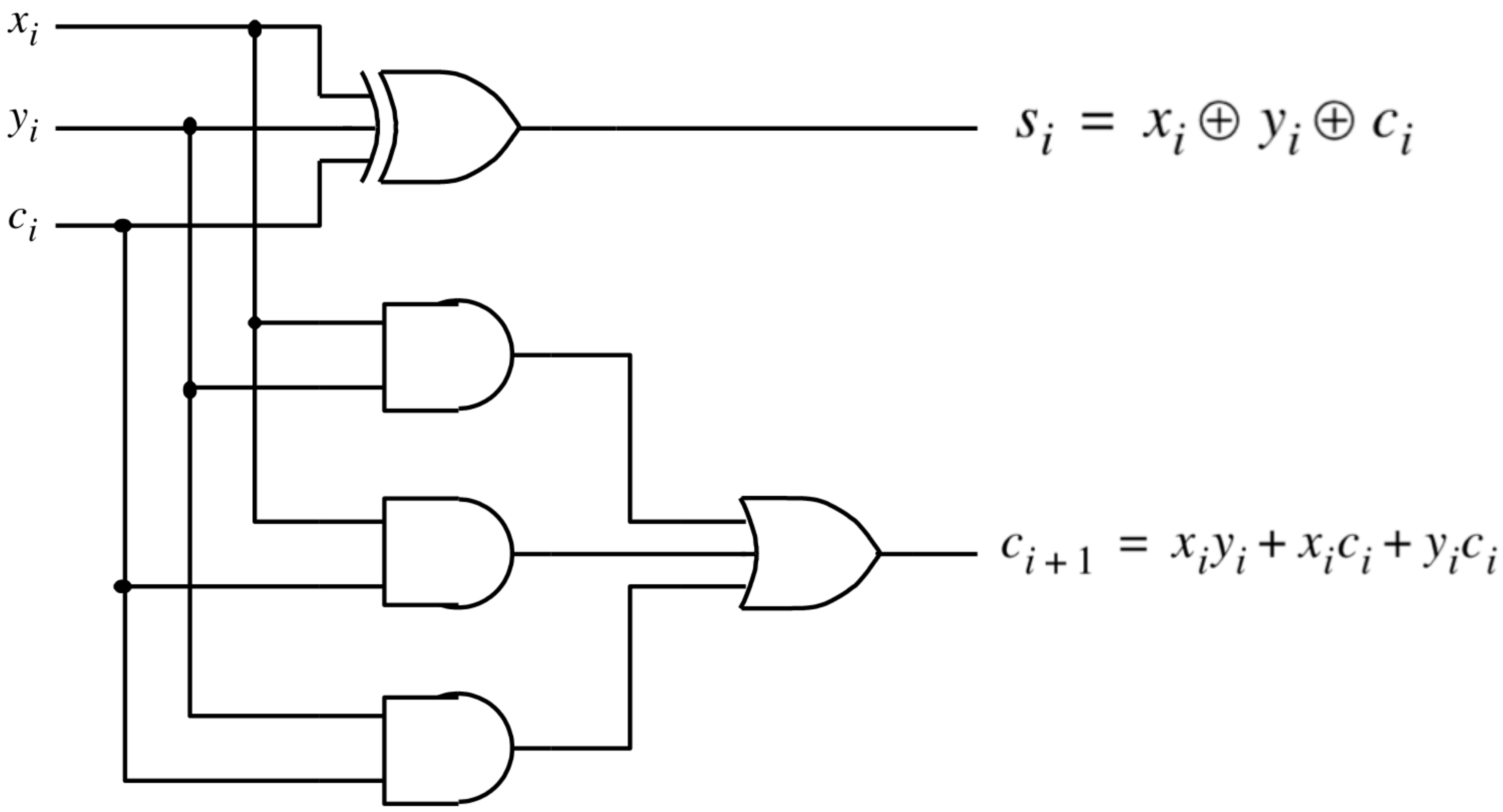
The goal is to evaluate very fast if the carry from the previous stage will be equal to 0 or 1.

Can we perform addition even faster?

The goal is to evaluate very fast if the carry from the previous stage will be equal to 0 or 1.

To accomplish this goal we will have to redesign the full-adder circuit yet again.

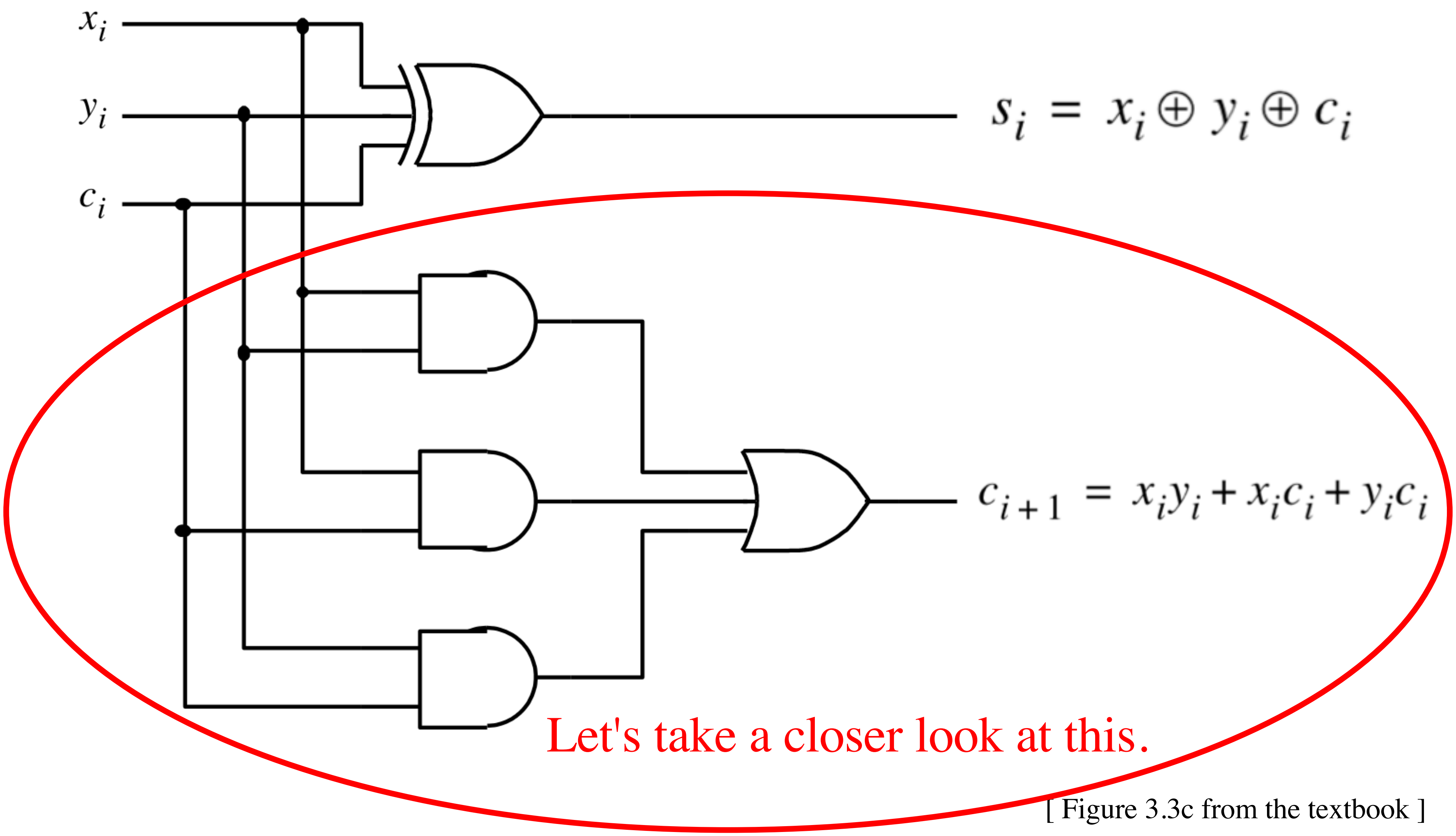
The Full-Adder Circuit



x_i
y_i
c_i
$s_i = x_i \oplus y_i \oplus c_i$
$c_{i+1} = x_i y_i + x_i c_i + y_i c_i$

[Figure 3.3c from the textbook]

The Full-Adder Circuit


x_i
y_i
c_i
$s_i = x_i \oplus y_i \oplus c_i$
$c_{i+1} = x_i y_i + x_i c_i + y_i c_i$
Let's take a closer look at this.

[Figure 3.3c from the textbook]

Decomposing the Carry Expression

$$c_{i+1} = x_i y_i + x_i c_i + y_i c_i$$

Decomposing the Carry Expression

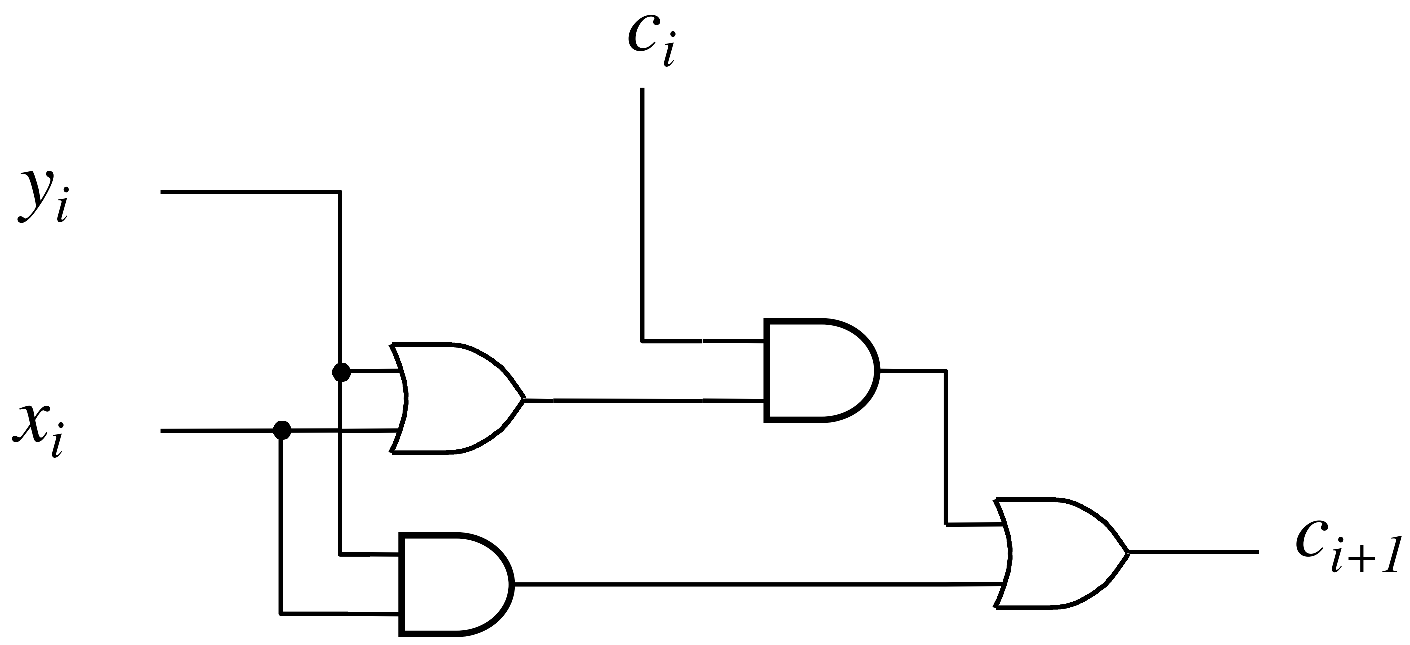
$$c_{i+1} = x_i y_i + x_i c_i + y_i c_i$$

$$c_{i+1} = x_i y_i + (x_i + y_i) c_i$$

Decomposing the Carry Expression

$$c_{i+1} = x_i y_i + x_i c_i + y_i c_i$$

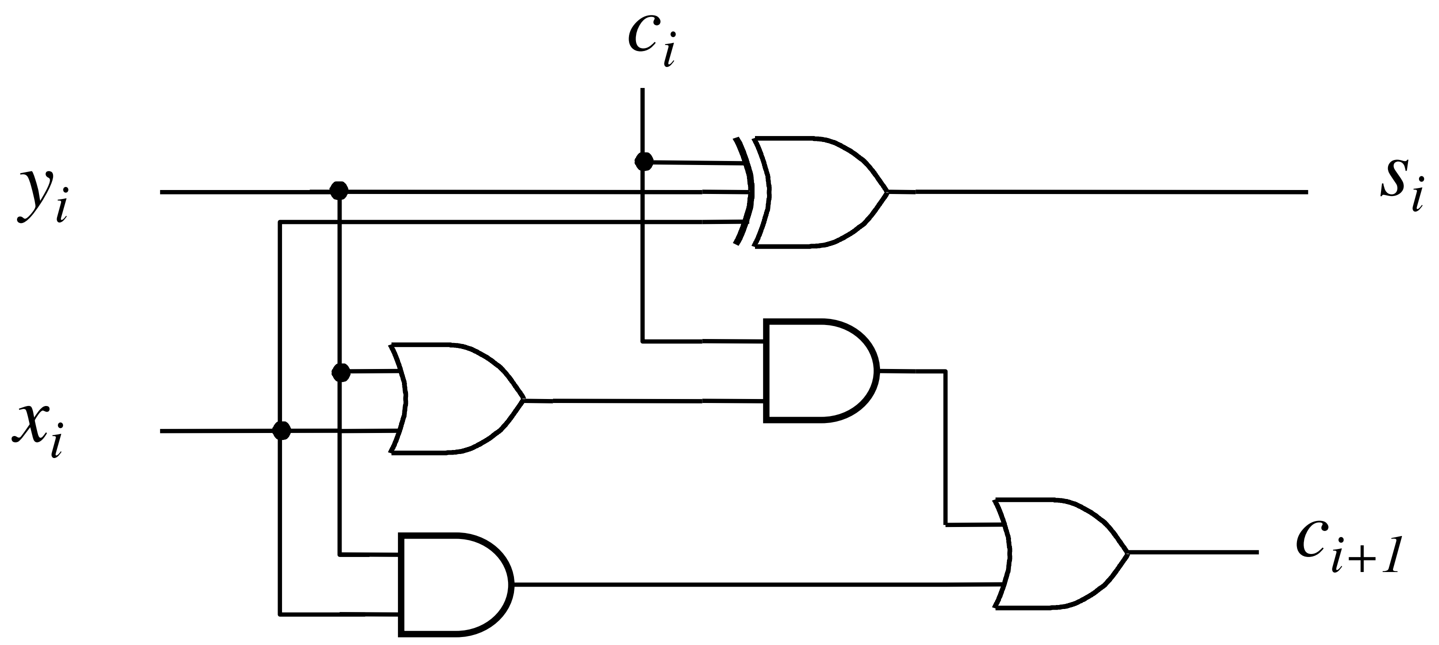
$$c_{i+1} = x_i y_i + (x_i + y_i) c_i$$



Another Way to Draw the Full-Adder Circuit

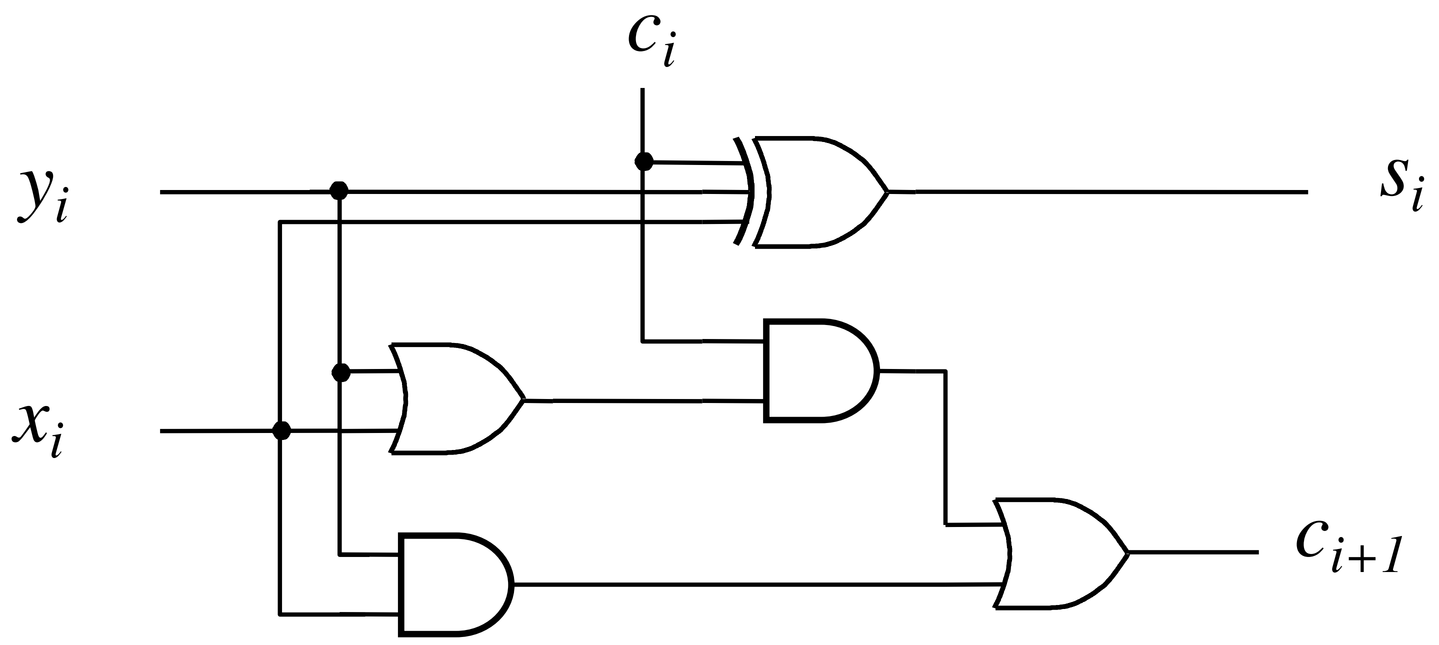
$$c_{i+1} = x_i\, y_i + x_i\, c_i + y_i\, c_i$$

$$c_{i+1} = x_i\, y_i + (x_i + y_i) c_i$$



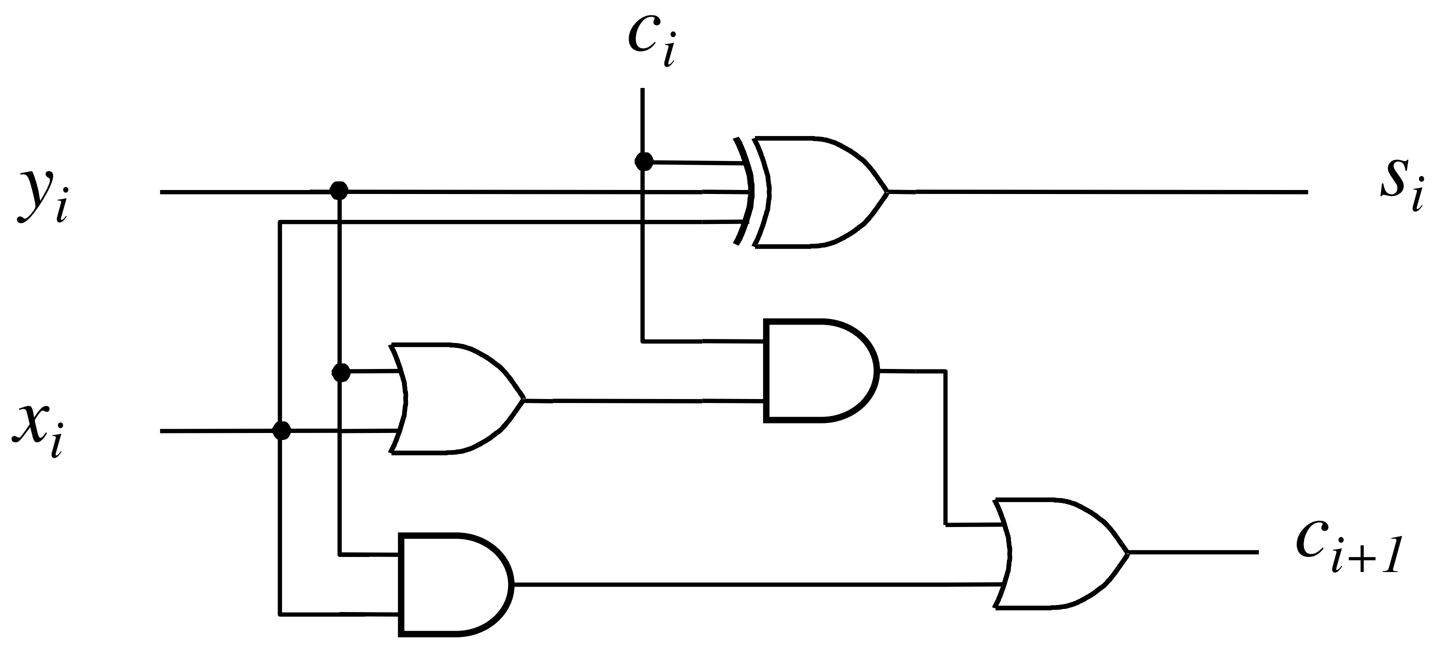
Another Way to Draw the Full-Adder Circuit

$$c_{i+1} = x_i y_i + (x_i + y_i) c_i$$



Another Way to Draw the Full-Adder Circuit

$$c_{i+1} = \underbrace{x_i y_i}_{g_i} + \underbrace{(x_i + y_i)}_{p_i} c_i$$

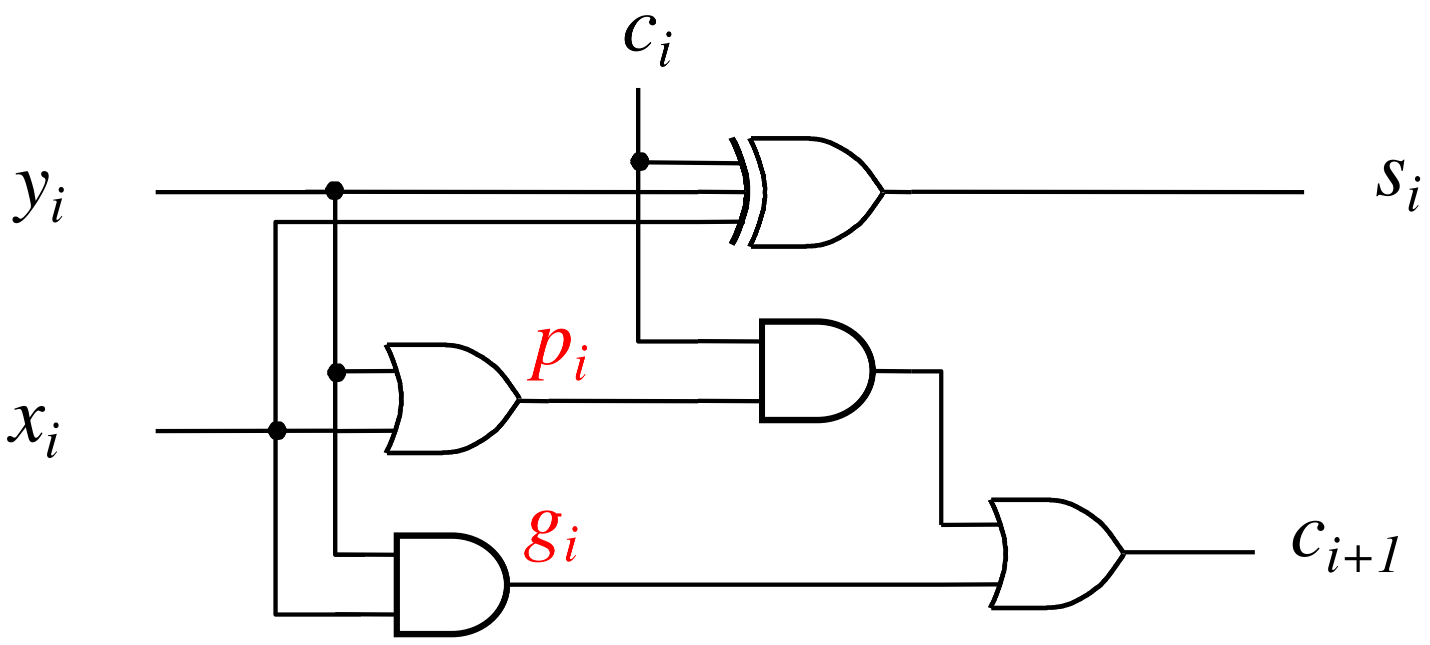


Another Way to Draw the Full-Adder Circuit

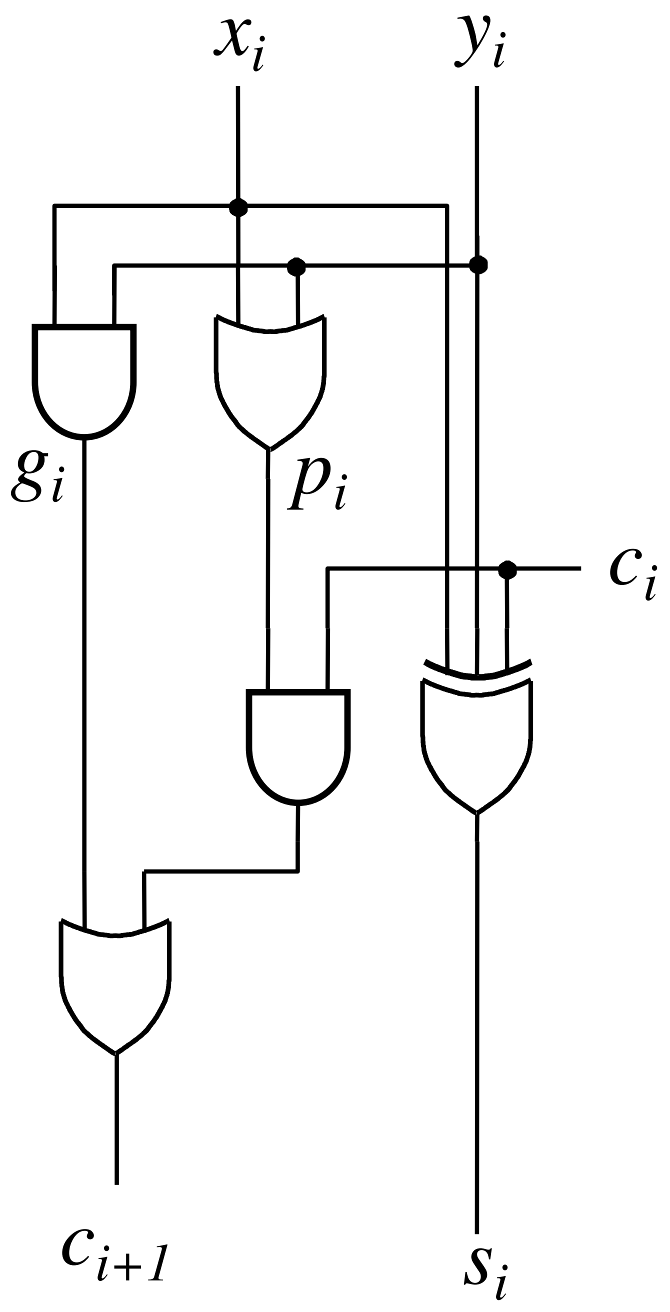
g - generate

p - propagate

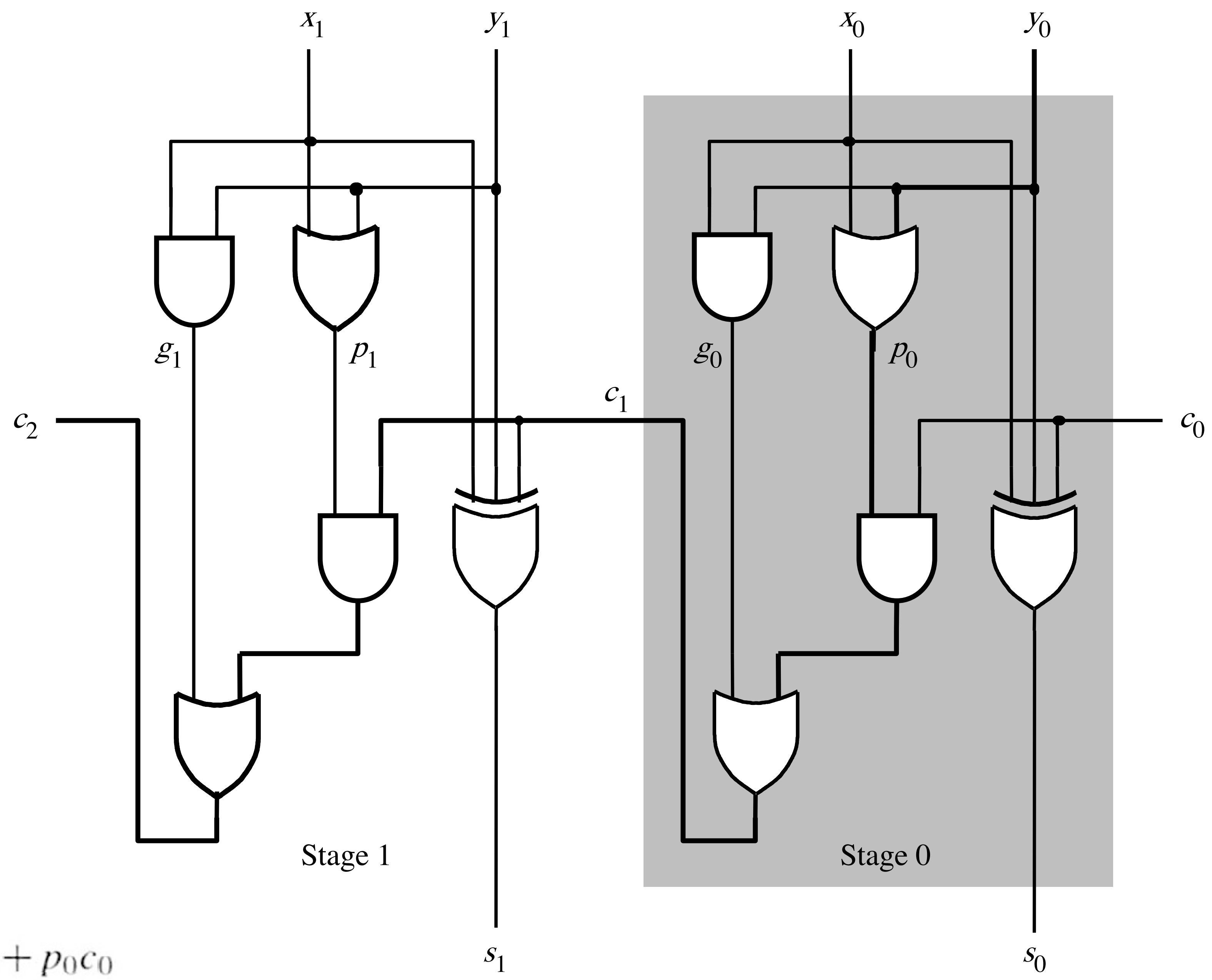
$$c_{i+1} = \underbrace{x_i y_i}_{g_i} + \underbrace{(x_i + y_i)}_{p_i} c_i$$



Yet Another Way to Draw It (Just Rotate It)



Now we can Build a Ripple-Carry Adder

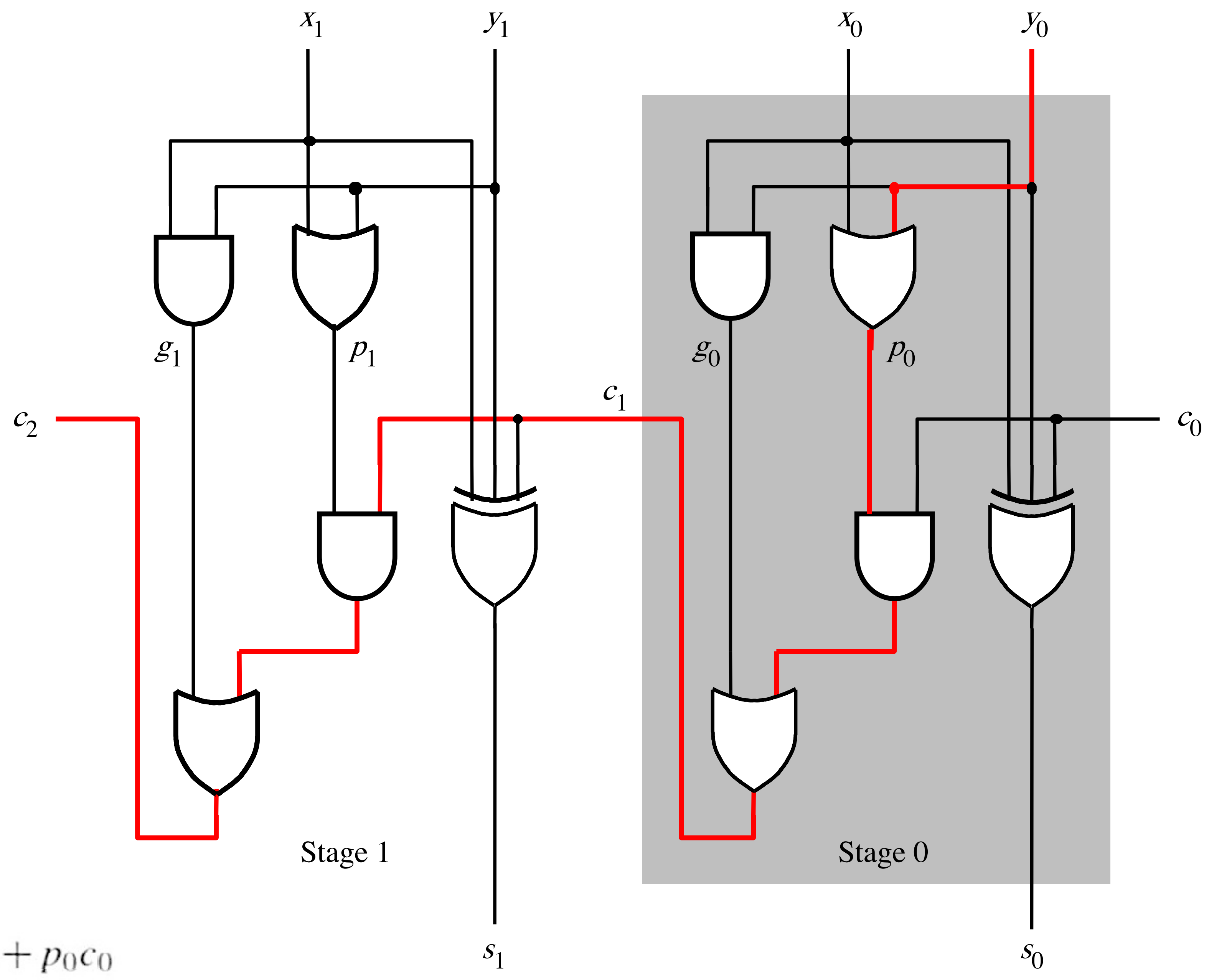


$$c_1 = g_0 + p_0 c_0$$

$$c_2 = g_1 + p_1 g_0 + p_1 p_0 c_0$$

[Figure 3.14 from the textbook]

Now we can Build a Ripple-Carry Adder

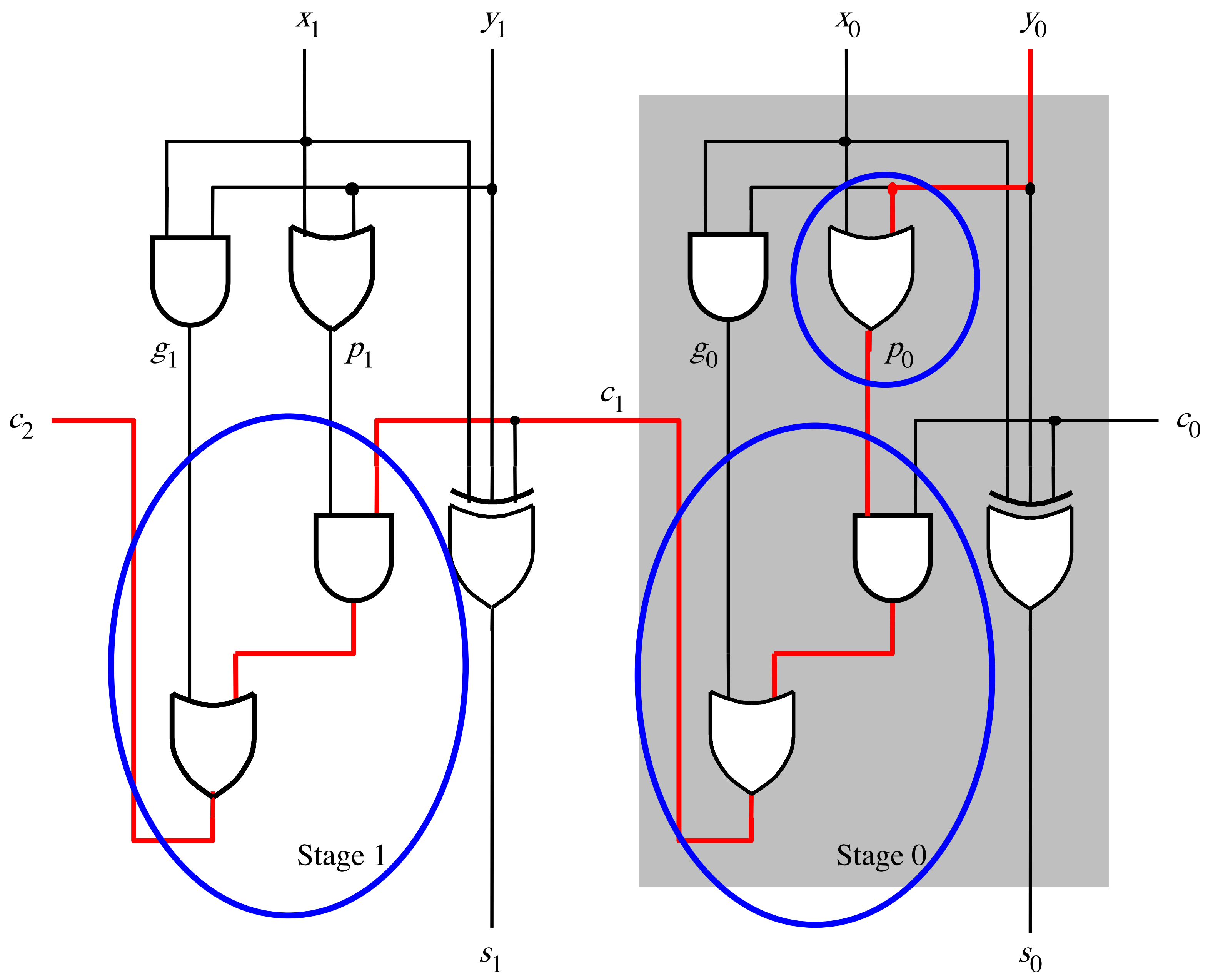


$$c_1 = g_0 + p_0 c_0$$

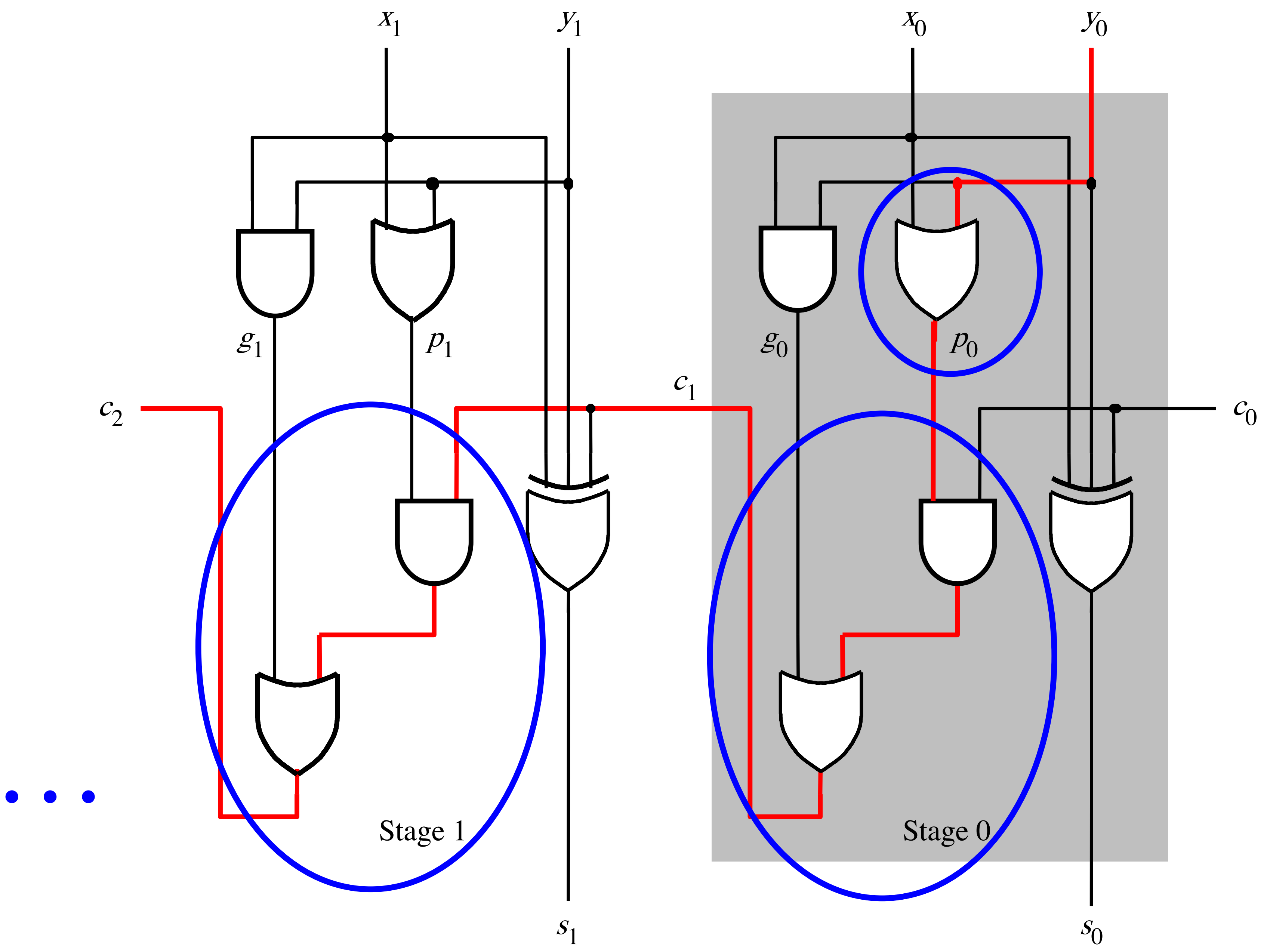
$$c_2 = g_1 + p_1 g_0 + p_1 p_0 c_0$$

[Figure 3.14 from the textbook]

2-bit ripple-carry adder: 5 gate delays (1+2+2)



n-bit ripple-carry adder: 2n+1 gate delays



n-bit Ripple-Carry Adder

- It takes 1 gate delay to generate all g_i and p_i signals
- +2 more gate delays to generate carry 1
- +2 more gate delay to generate carry 2

  …
- +2 more gate delay to generate carry n
- Thus, the total delay through an
  n-bit ripple-carry adder is 2n+1 gate delays!

n-bit Ripple-Carry Adder

- **It takes 1 gate delay to generate all g_i and p_i signals**
- **+2 more gate delays to generate carry 1**
- **+2 more gate delay to generate carry 2**

  …
- **+2 more gate delay to generate carry n**
- **Thus, the total delay through an n-bit ripple-carry adder is 2n+1 gate delays!**

This is slower by 1 than the original design?!

A carry-lookahead adder

Decomposing the Carry Expression

$$c_{i+1} = x_i \, y_i + x_i \, c_i + y_i \, c_i$$

Decomposing the Carry Expression

$$c_{i+1} = x_i y_i + x_i c_i + y_i c_i$$

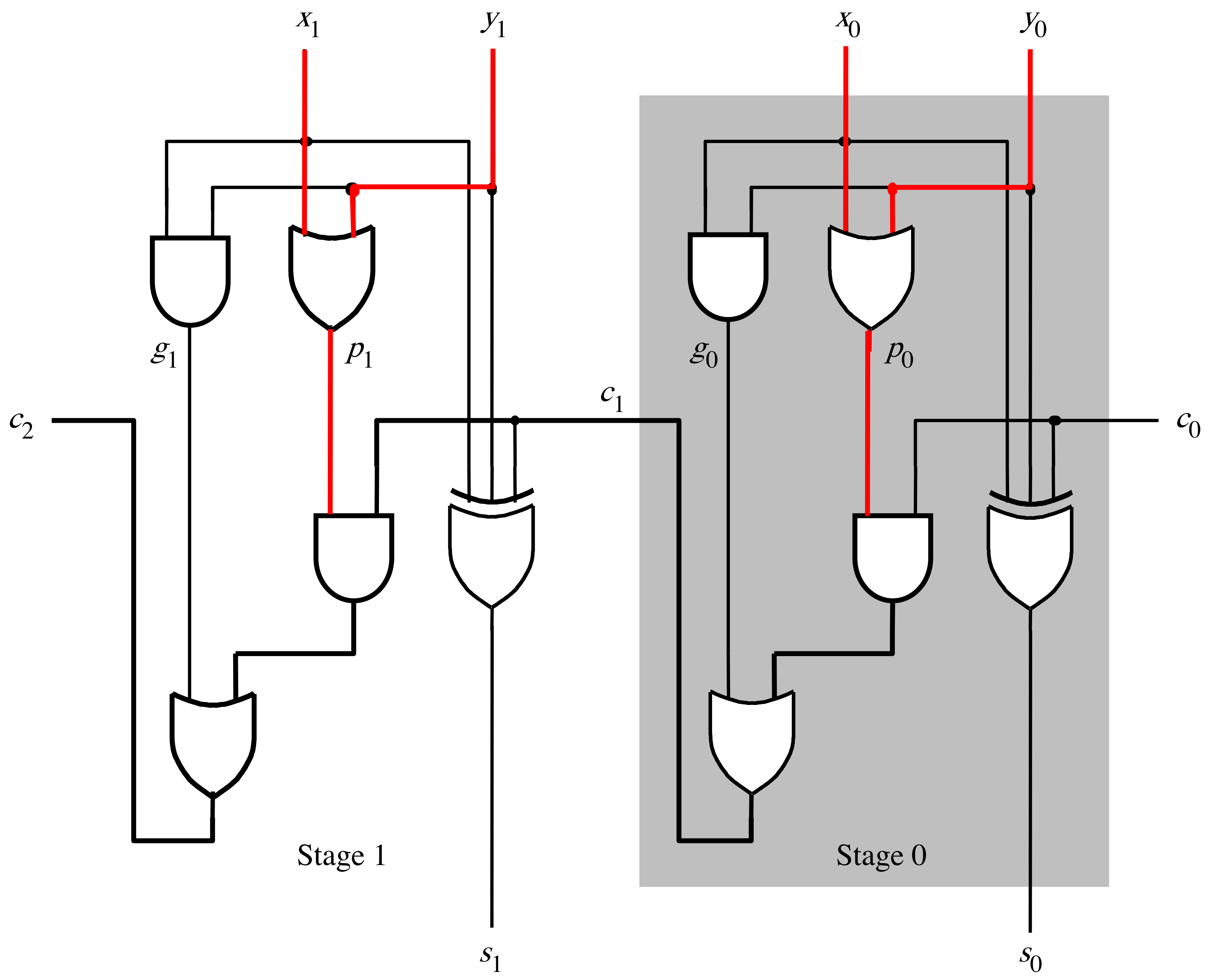
$$c_{i+1} = \underbrace{x_i y_i}_{g_i} + \underbrace{(x_i + y_i)}_{p_i} c_i$$

Decomposing the Carry Expression

$$c_{i+1} = x_i y_i + x_i c_i + y_i c_i$$

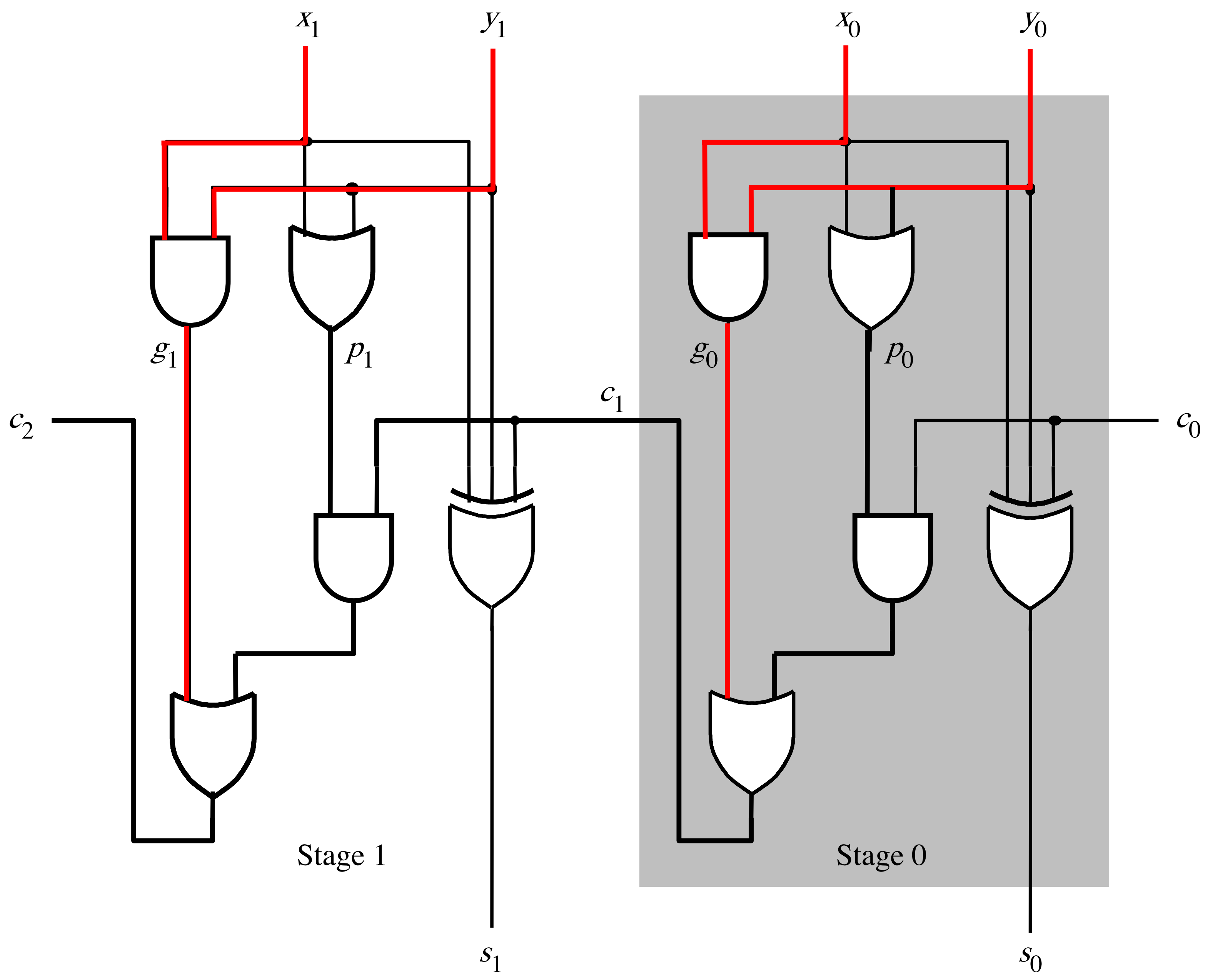
$$c_{i+1} = \underbrace{x_i y_i}_{\substack{g_i \\ \text{(1 gate delay)}}} + \underbrace{(x_i + y_i)}_{\substack{p_i \\ \text{(1 gate delay)}}} c_i$$

It takes 1 gate delay to compute all p_i signals



[Figure 3.14 from the textbook]

It takes 1 gate delay to compute all g_i signals



[Figure 3.14 from the textbook]

Decomposing the Carry Expression

$$c_{i+1} = x_i y_i + x_i c_i + y_i c_i$$

$$c_{i+1} = \underbrace{x_i y_i}_{g_i} + \underbrace{(x_i + y_i)}_{p_i} c_i$$

Decomposing the Carry Expression

$$c_{i+1} = x_i y_i + x_i c_i + y_i c_i$$

$$c_{i+1} = \underbrace{x_i y_i}_{g_i} + \underbrace{(x_i + y_i)}_{p_i} c_i$$

$$c_{i+1} = g_i + p_i c_i$$

Decomposing the Carry Expression

$$c_{i+1} = x_i\, y_i + x_i\, c_i + y_i\, c_i$$

$$c_{i+1} = \underbrace{x_i\, y_i}_{g_i} + \underbrace{(x_i + y_i)}_{p_i} c_i$$

$$c_{i+1} = g_i + p_i\, c_i$$

recursive expansion of c_i

$$c_{i+1} = g_i + p_i \boxed{(g_{i-1} + p_{i-1}\, c_{i-1})}$$

Decomposing the Carry Expression

$$c_{i+1} = x_i y_i + x_i c_i + y_i c_i$$

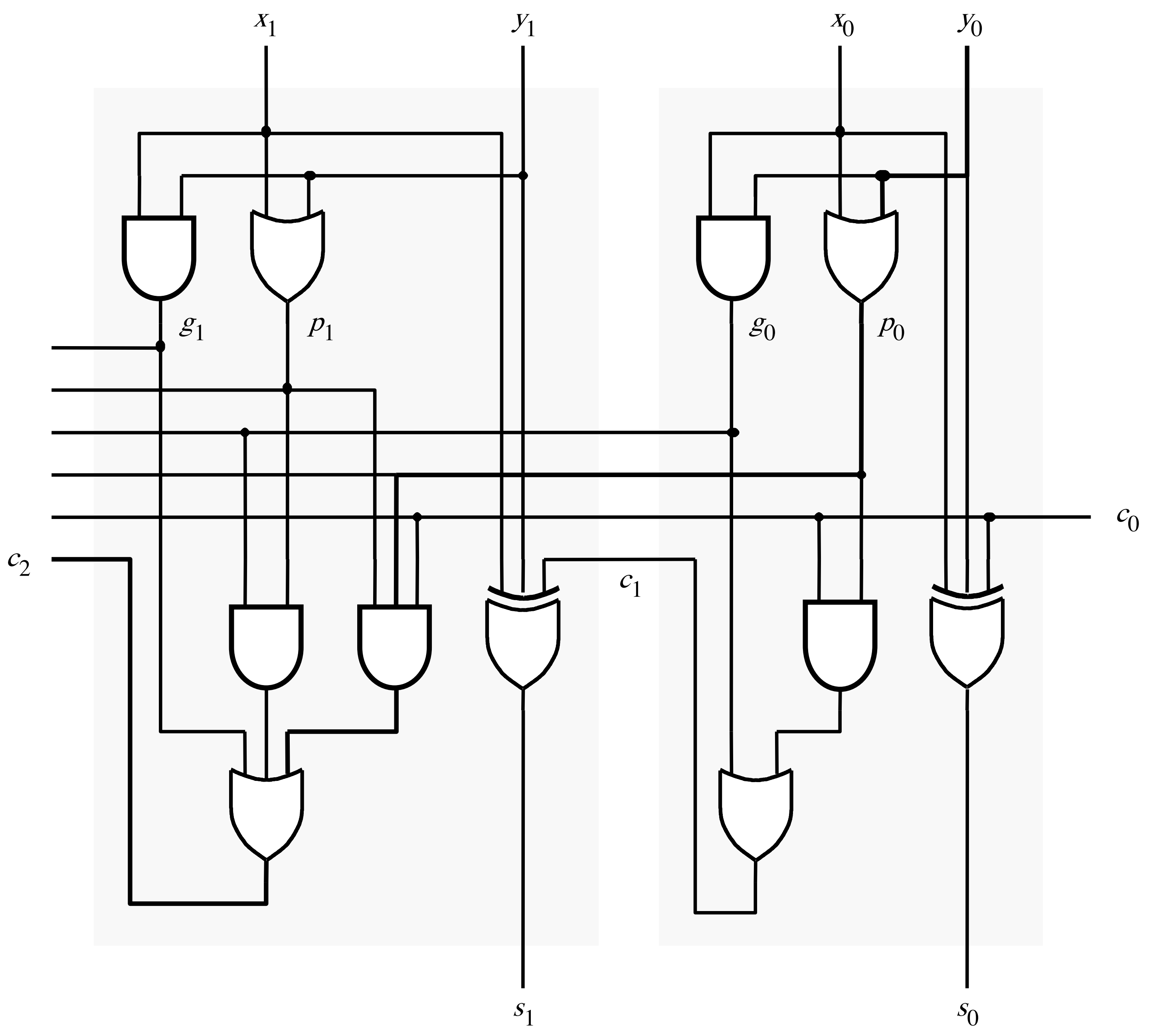
$$c_{i+1} = \underbrace{x_i y_i}_{g_i} + \underbrace{(x_i + y_i)}_{p_i} c_i$$

$$c_{i+1} = g_i + p_i c_i$$

$$c_{i+1} = g_i + p_i (g_{i-1} + p_{i-1} c_{i-1})$$

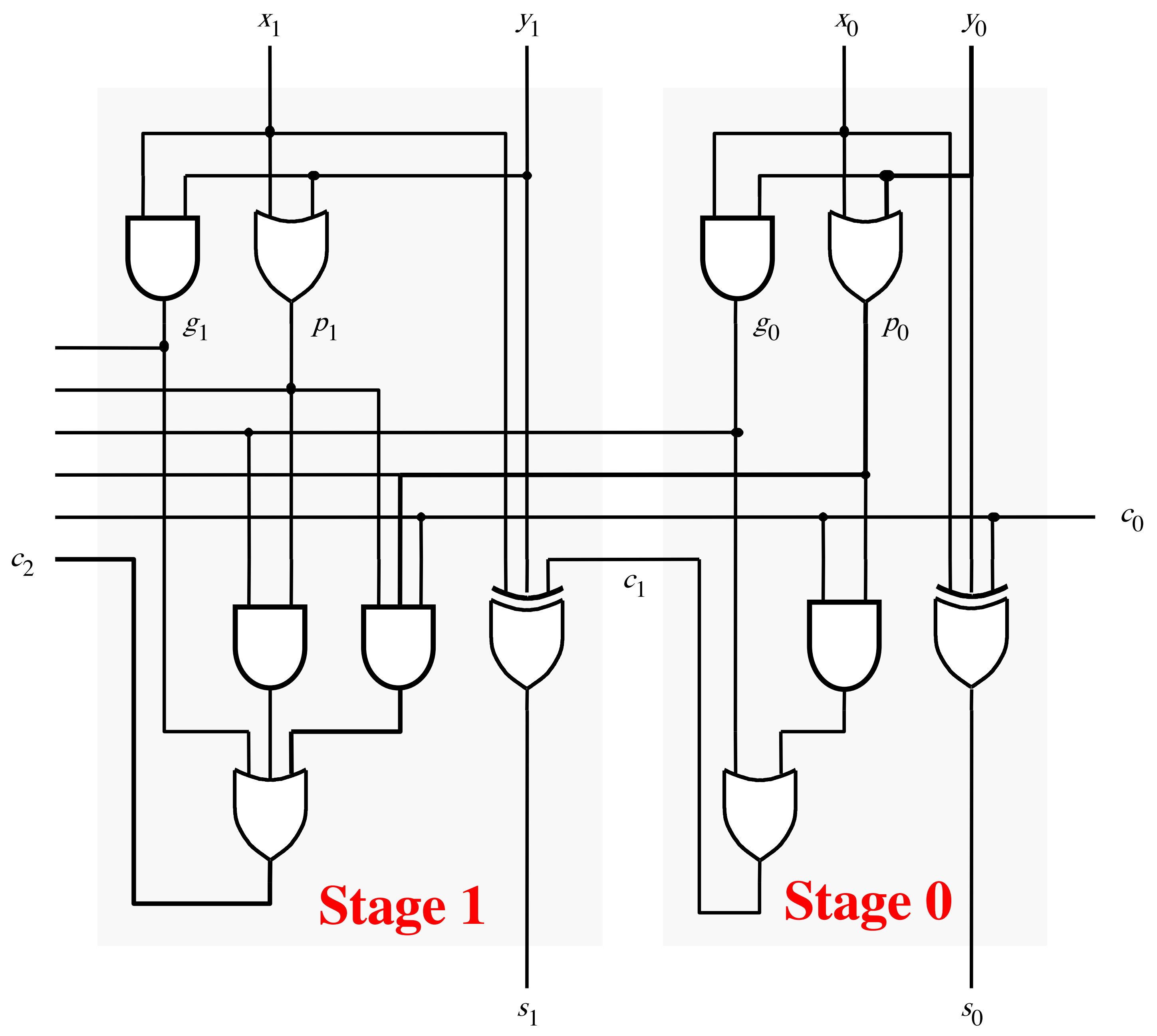
$$c_{i+1} = g_i + p_i g_{i-1} + p_i p_{i-1} c_{i-1}$$

Now we can Build a Carry-Lookahead Adder



[Figure 3.15 from the textbook]

The first two stages of a carry-lookahead adder



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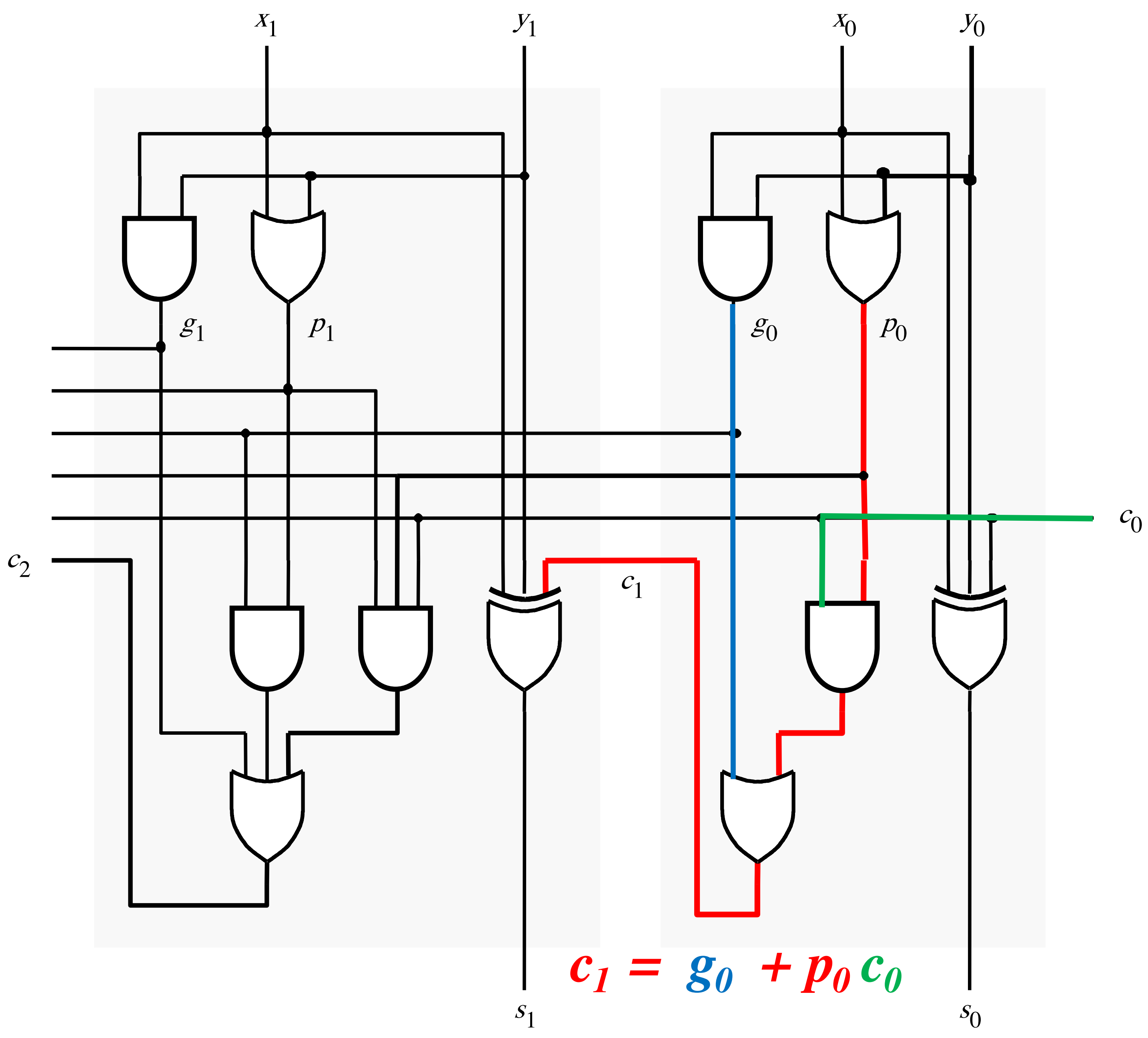
[Figure 3.15 from the textbook]

Carry for the first stage

$$c_1 = g_0 + p_0 c_0$$

Carry for the first stage

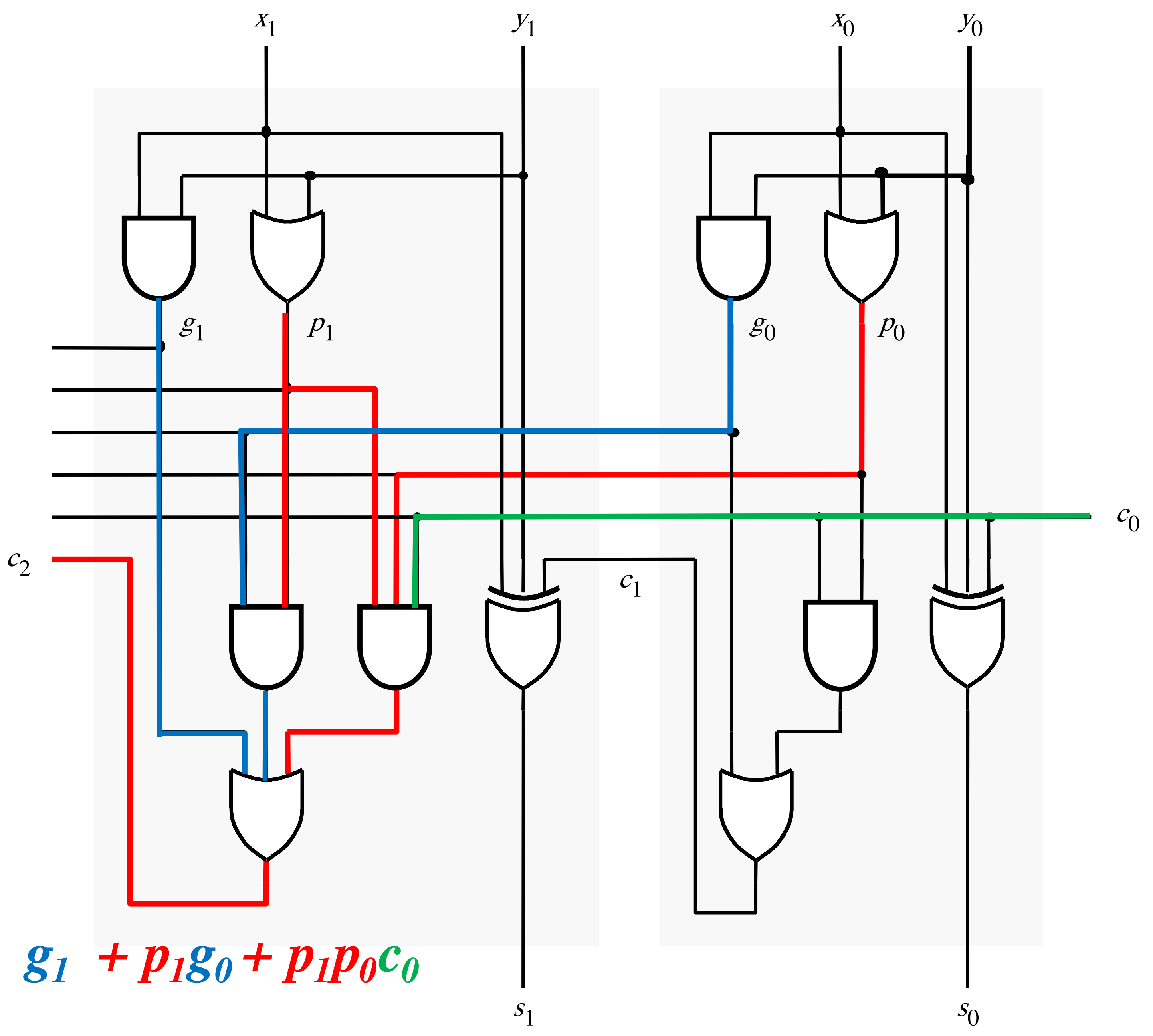


$c_1 = g_0 + p_0 c_0$

Carry for the second stage

$$c_2 = g_1 + p_1 g_0 + p_1 p_0 c_0$$

Carry for the second stage



$c_2 = g_1 + p_1 g_0 + p_1 p_0 c_0$

Carry for the first two stages

$$c_1 = g_0 + p_0 c_0$$

$$c_2 = g_1 + p_1 g_0 + p_1 p_0 c_0$$

Carry for the first two stages

$$c_1 = g_0 + p_0 c_0$$

$$c_2 = g_1 + \underline{p_1} g_0 + \underline{p_1} p_0 c_0$$

Carry for the first two stages

$$c_1 = g_0 + p_0 c_0$$

$$c_2 = g_1 + \underline{p_1} g_0 + \underline{p_1} p_0 c_0$$

$$= g_1 + p_1 (\underbrace{g_0 + p_0 c_0}_{c_1})$$

Carry for the first two stages

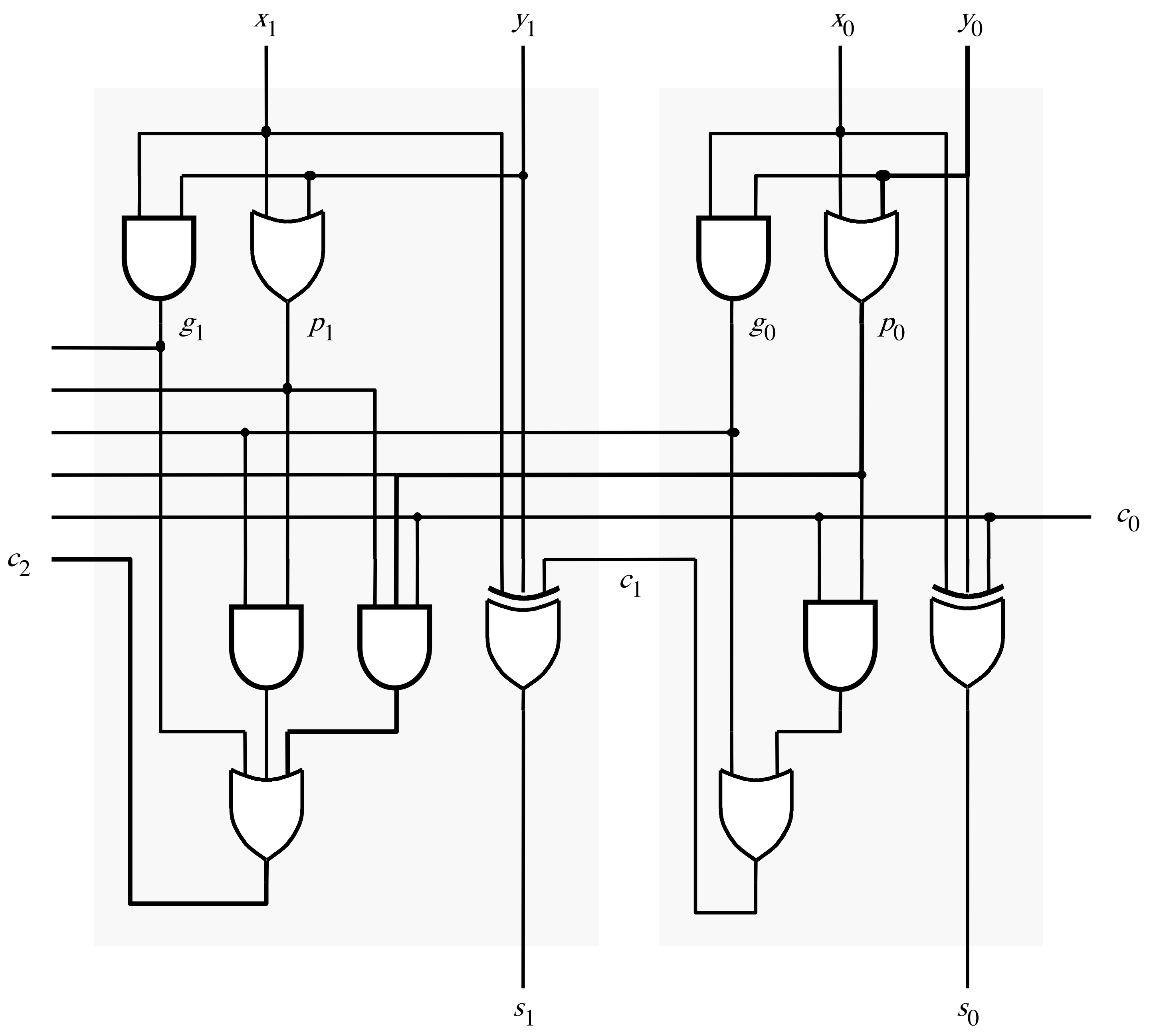
$$c_1 = g_0 + p_0 c_0$$

$$c_2 = g_1 + p_1 g_0 + p_1 p_0 c_0$$

$$= g_1 + p_1 \underbrace{(g_0 + p_0 c_0)}_{c_1}$$

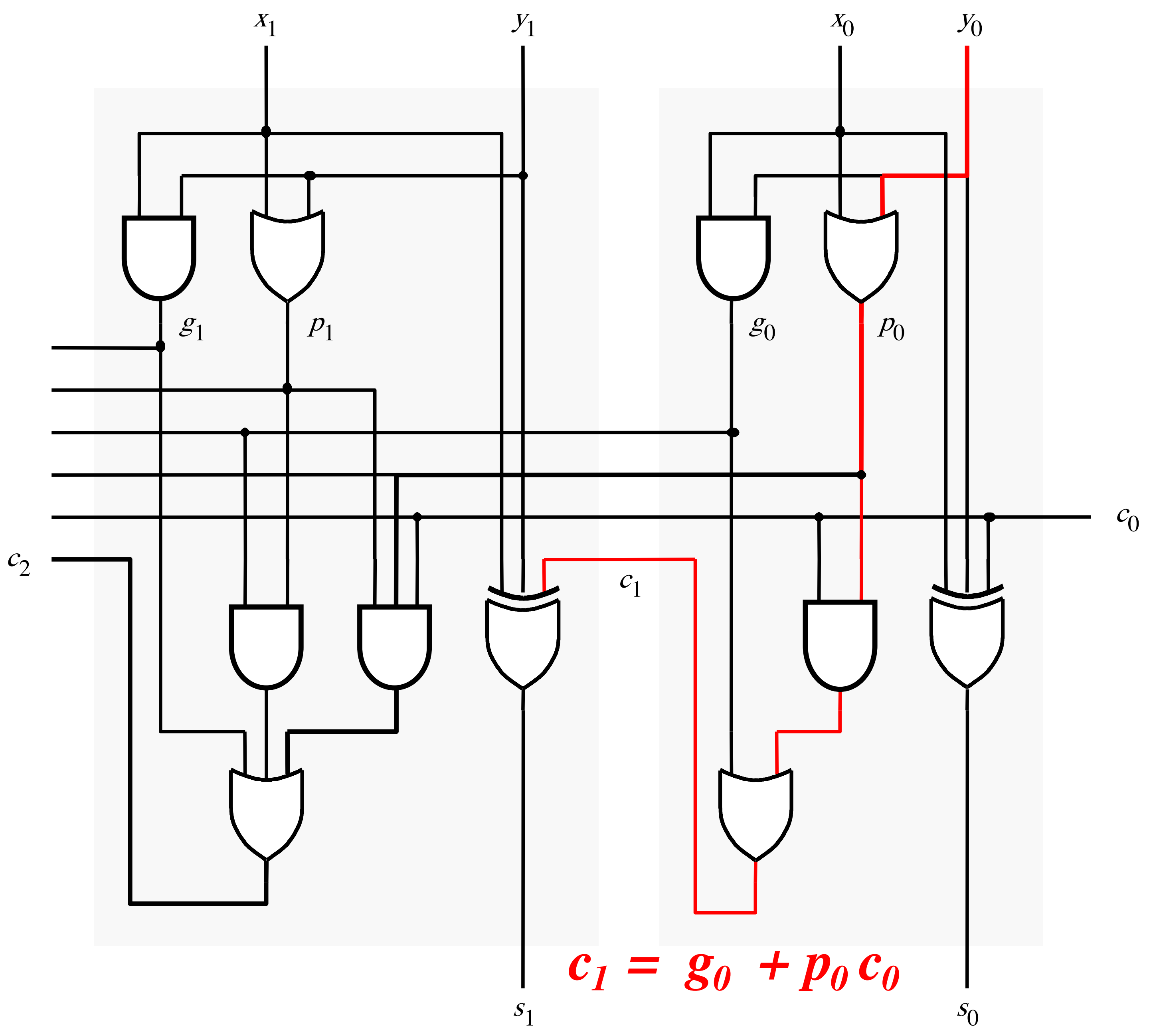
$$= g_1 + p_1 c_1$$

The first two stages of a carry-lookahead adder



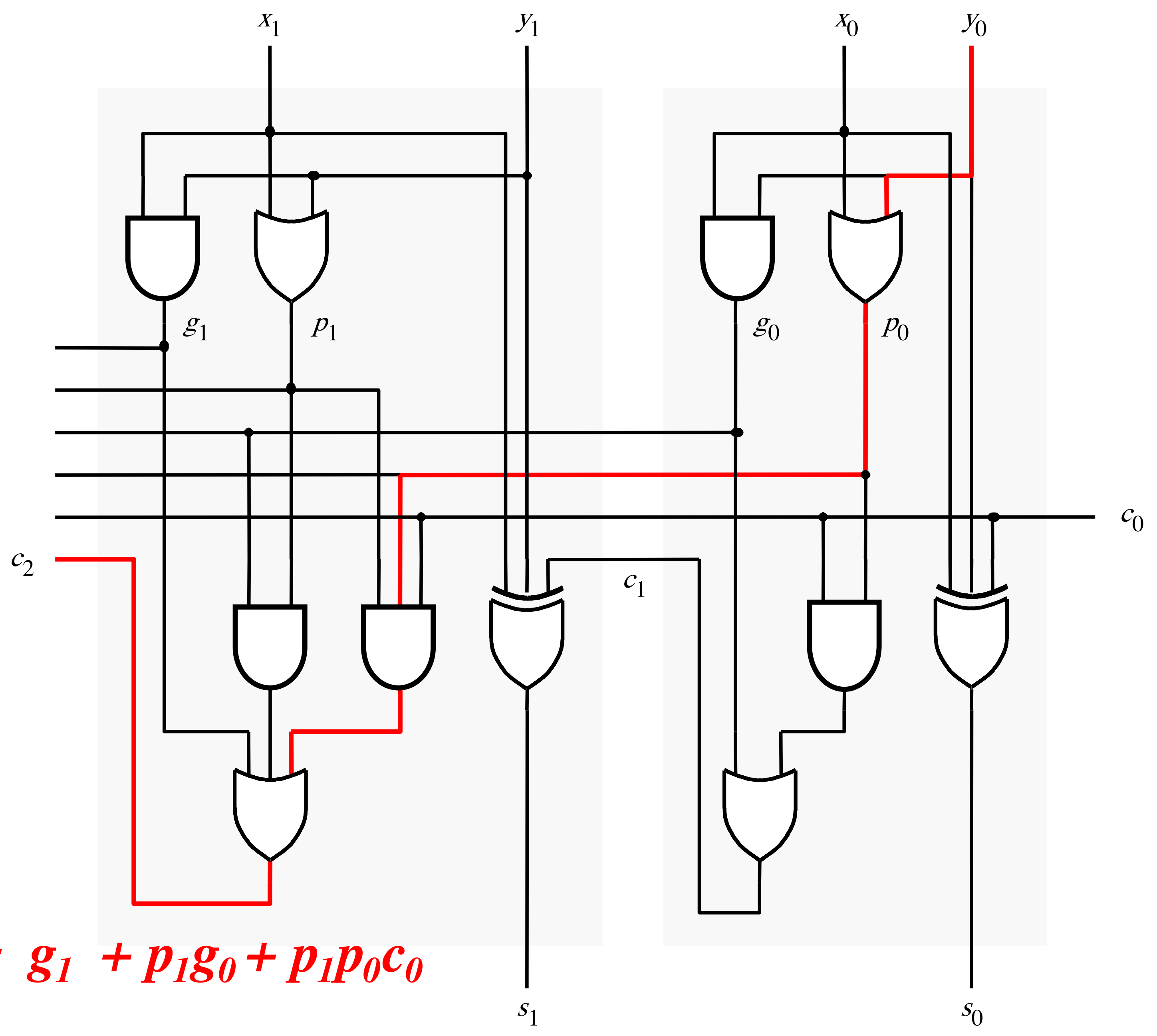
[Figure 3.15 from the textbook]

It takes 3 gate delays to generate c_1



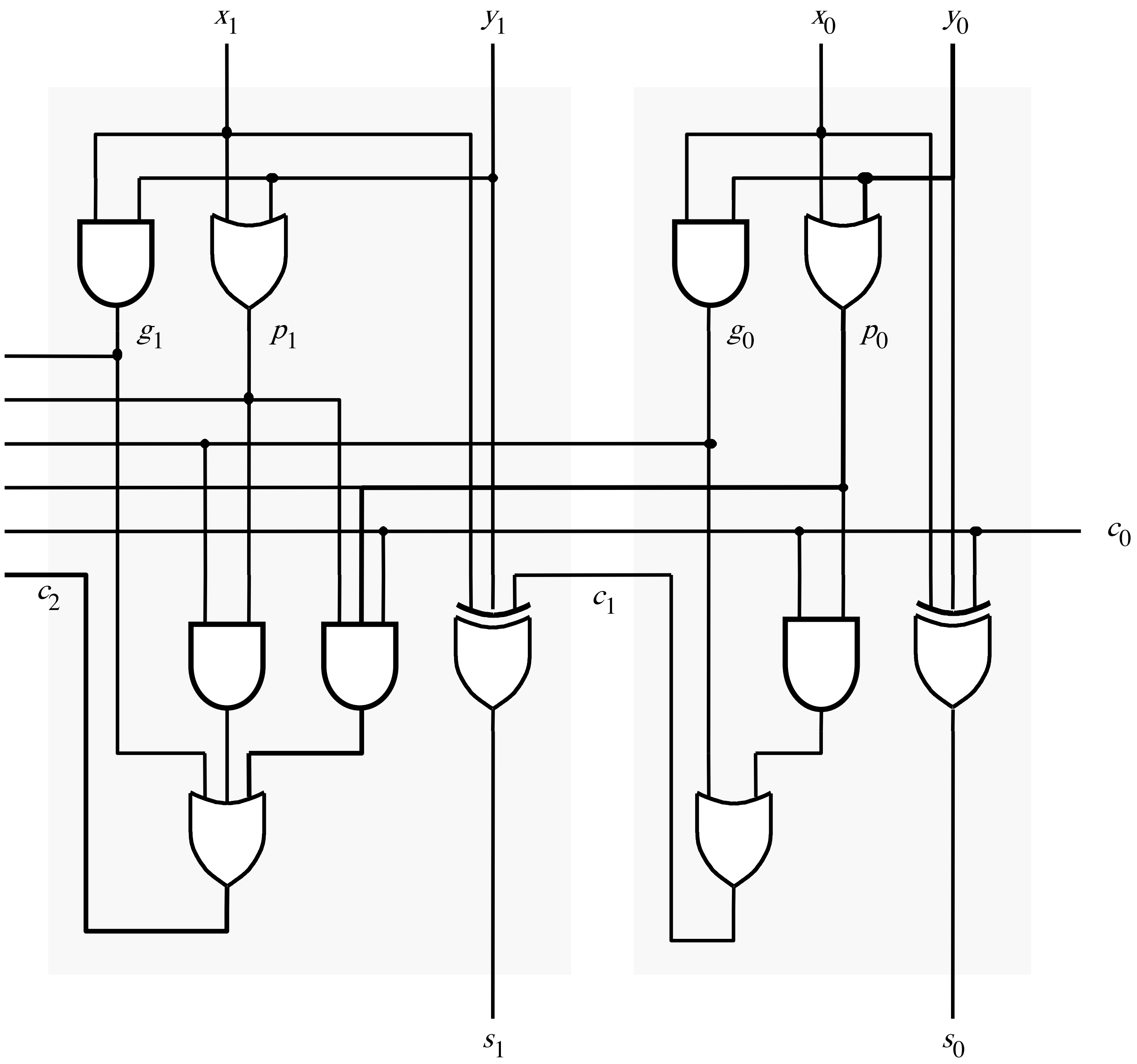
$c_1 = g_0 + p_0 c_0$

It takes 3 gate delays to generate c_2

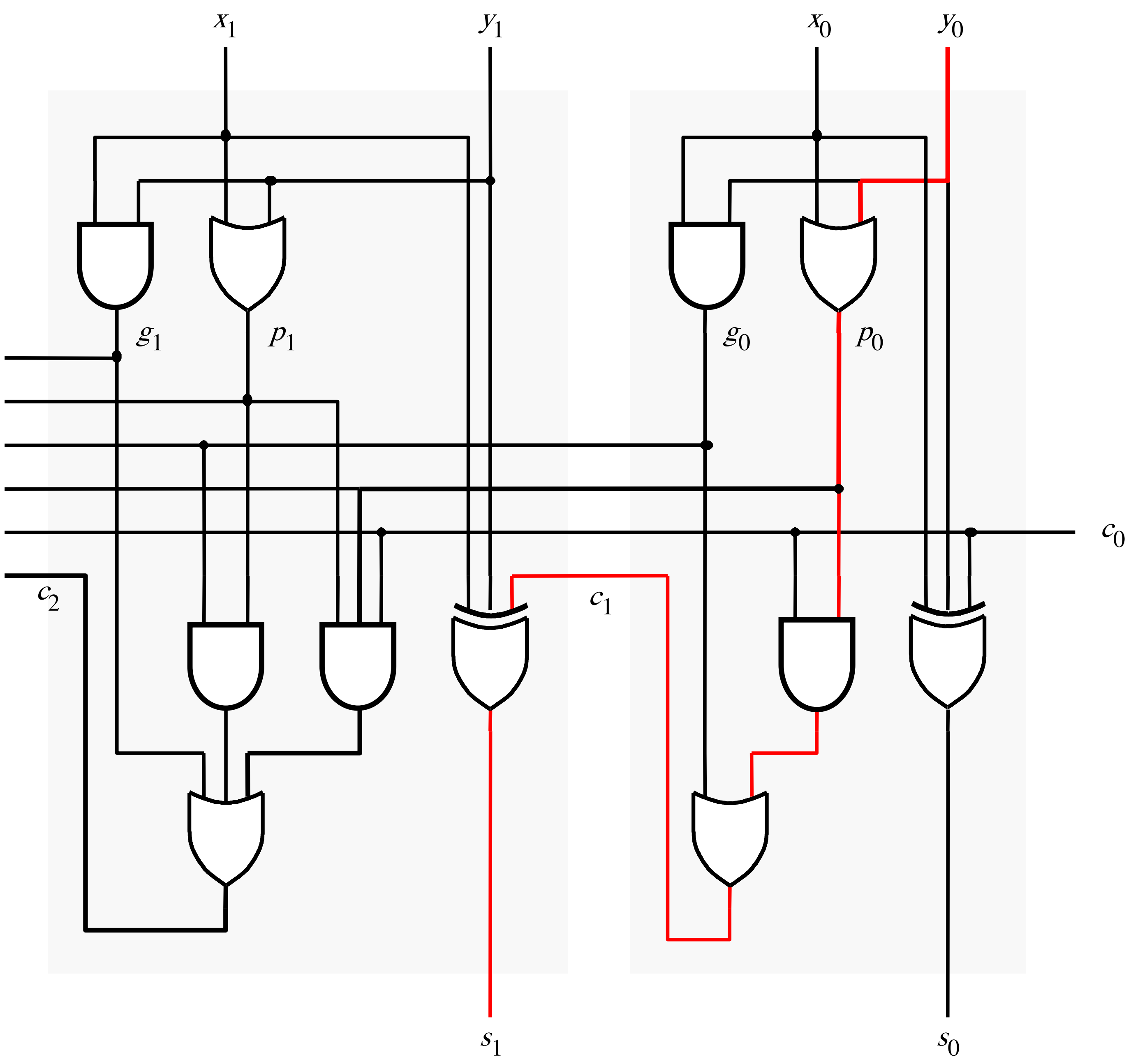


$c_2 = g_1 + p_1g_0 + p_1p_0c_0$

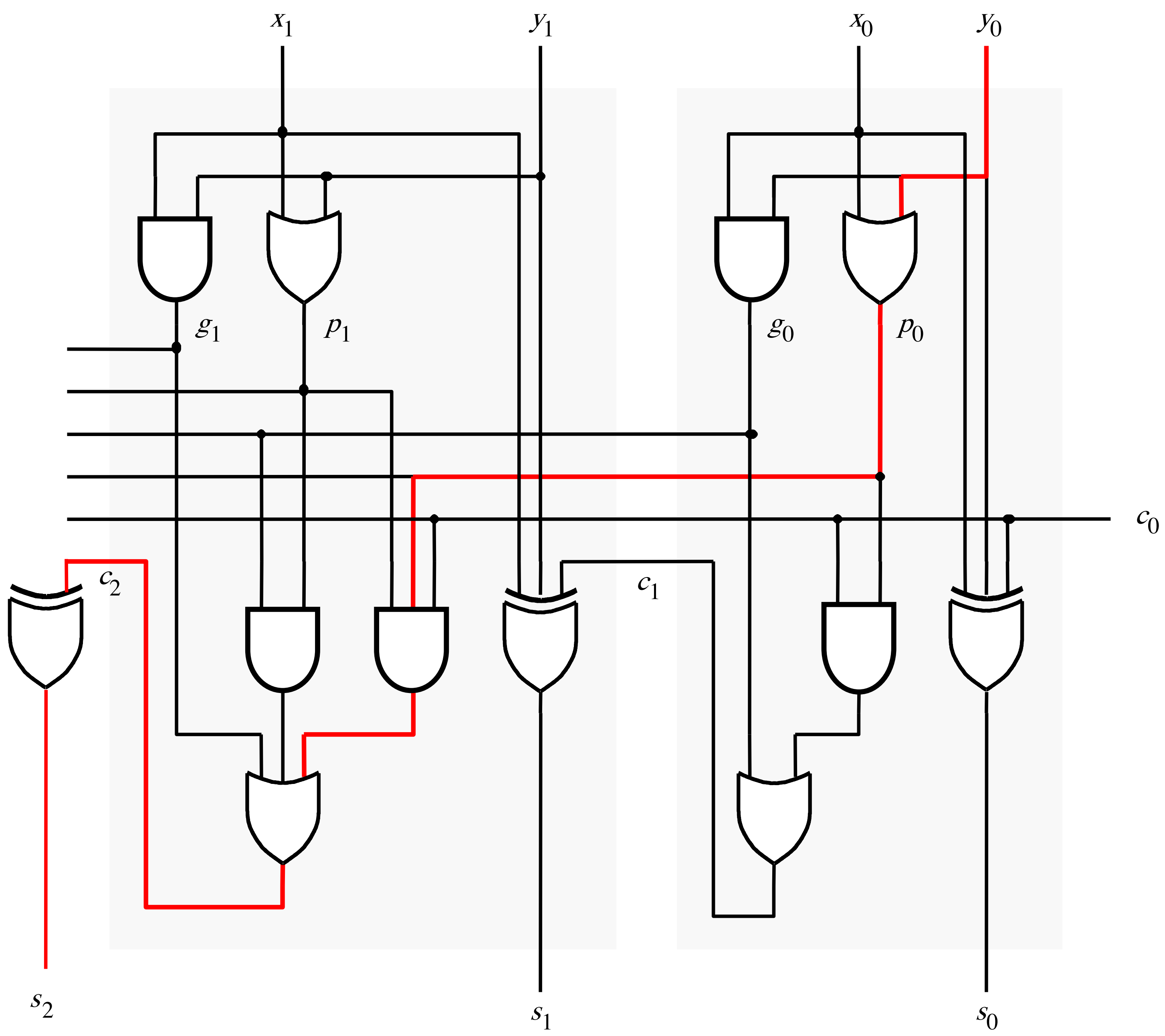
The first two stages of a carry-lookahead adder



It takes 4 gate delays to generate s_1



It takes 4 gate delays to generate s_2



N-bit Carry-Lookahead Adder

- It takes 1 gate delay to generate all g_i and p_i signals
- It takes 2 more gate delays to generate all carry signals
- It takes 1 more gate delay to generate all sum bits
- Thus, the total delay through an
  n-bit carry-lookahead adder is only 4 gate delays!

Expanding the Carry Expression

$$c_{i+1} = g_i + p_i c_i$$

$$c_1 = g_0 + p_0 c_0$$

$$c_2 = g_1 + p_1 g_0 + p_1 p_0 c_0$$

$$c_3 = g_2 + p_2 g_1 + p_2 p_1 g_0 + p_2 p_1 p_0 c_0$$

$$\cdots$$

$$\begin{aligned} c_8 = {} & g_7 + p_7 g_6 + p_7 p_6 g_5 + p_7 p_6 p_5 g_4 \\ & + p_7 p_6 p_5 p_4 g_3 + p_7 p_6 p_5 p_4 p_3 g_2 \\ & + p_7 p_6 p_5 p_4 p_3 p_2 g_1 + p_7 p_6 p_5 p_4 p_3 p_2 p_1 g_0 \\ & + p_7 p_6 p_5 p_4 p_3 p_2 p_1 p_0 c_0 \end{aligned}$$

Expanding the Carry Expression

$$c_{i+1} = g_i + p_i c_i$$

$$c_1 = g_0 + p_0 c_0$$

$$c_2 = g_1 + p_1 g_0 + p_1 p_0 c_0$$

$$c_3 = g_2 + p_2 g_1 + p_2 p_1 g_0 + p_2 p_1 p_0 c_0$$

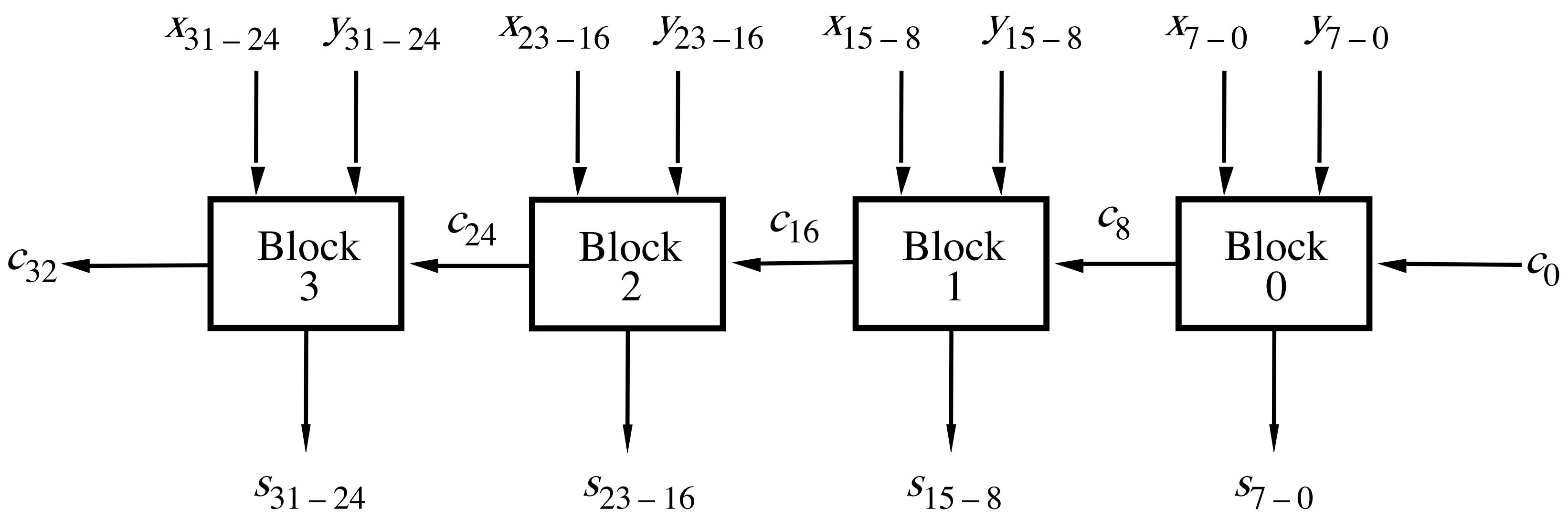
$$\cdots$$

$$\begin{aligned} c_8 = {} & g_7 + p_7 g_6 + p_7 p_6 g_5 + p_7 p_6 p_5 g_4 \\ & + p_7 p_6 p_5 p_4 g_3 + p_7 p_6 p_5 p_4 p_3 g_2 \\ & + p_7 p_6 p_5 p_4 p_3 p_2 g_1 + p_7 p_6 p_5 p_4 p_3 p_2 p_1 g_0 \\ & + p_7 p_6 p_5 p_4 p_3 p_2 p_1 p_0 c_0 \end{aligned}$$

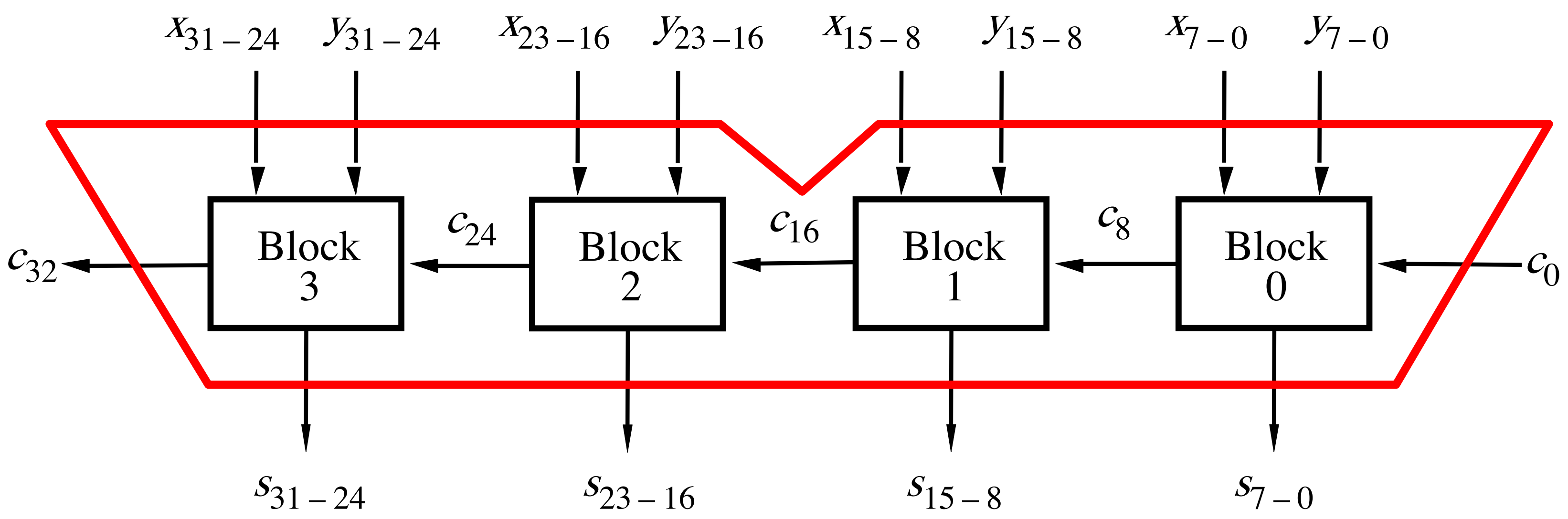
Even this takes only 3 gate delays

A hierarchical carry-lookahead adder with ripple-carry between blocks

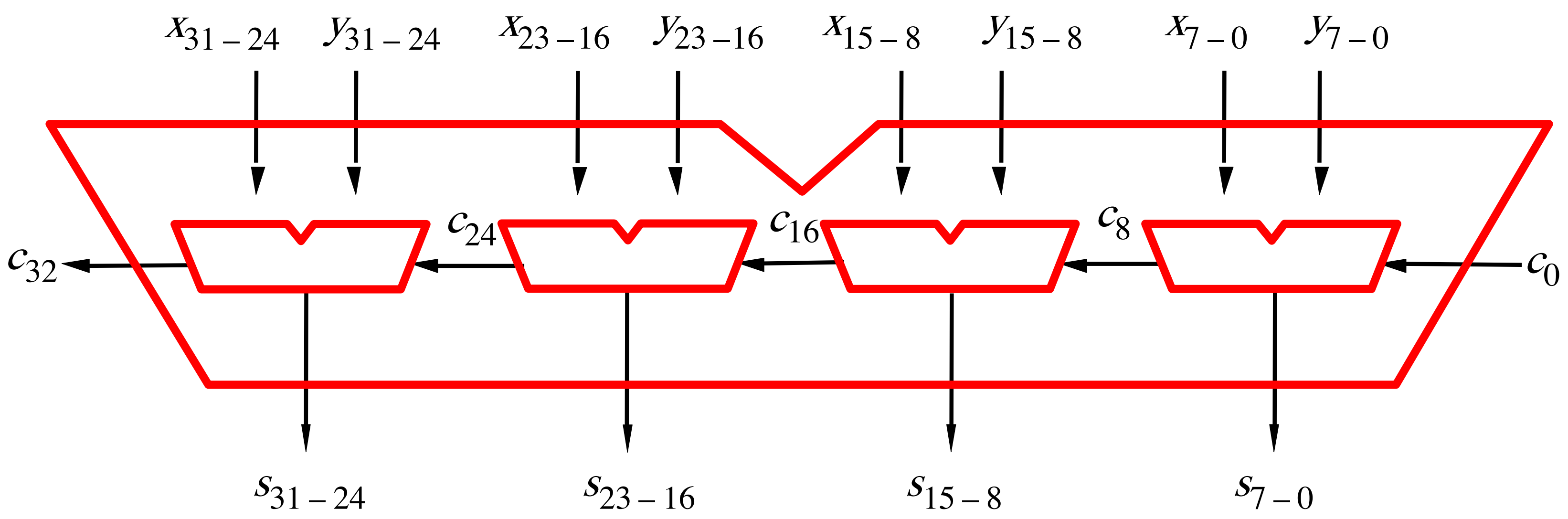
A hierarchical carry-lookahead adder with ripple-carry between blocks



A hierarchical carry-lookahead adder with ripple-carry between blocks

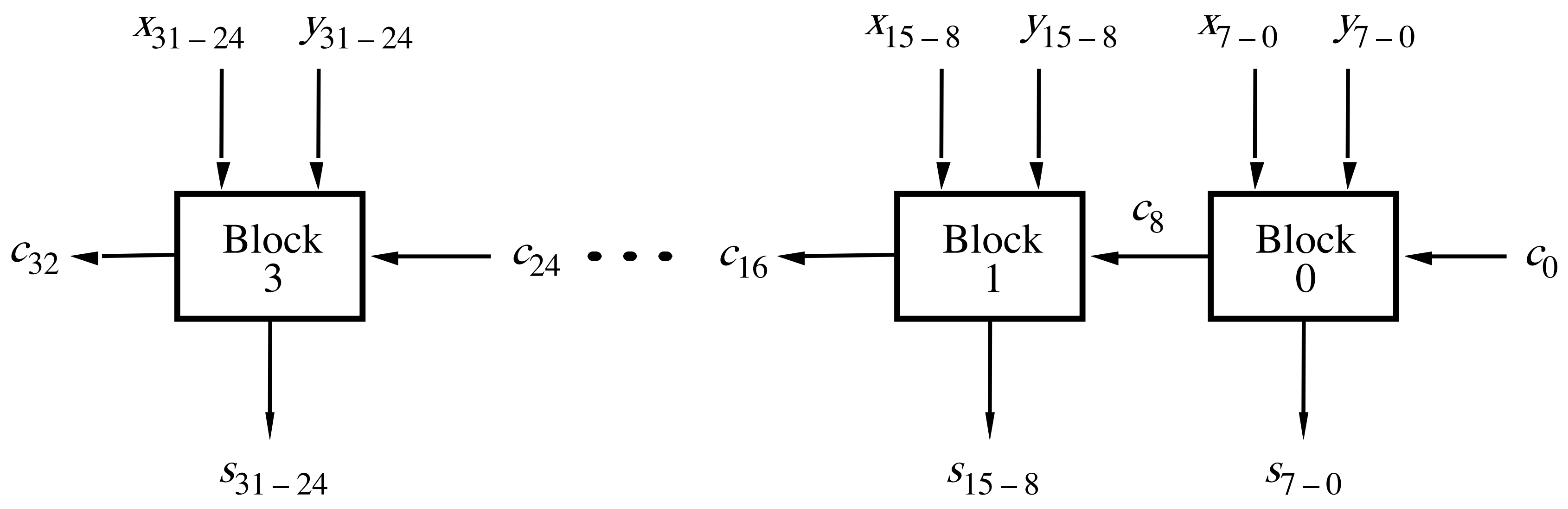


A hierarchical carry-lookahead adder with ripple-carry between blocks



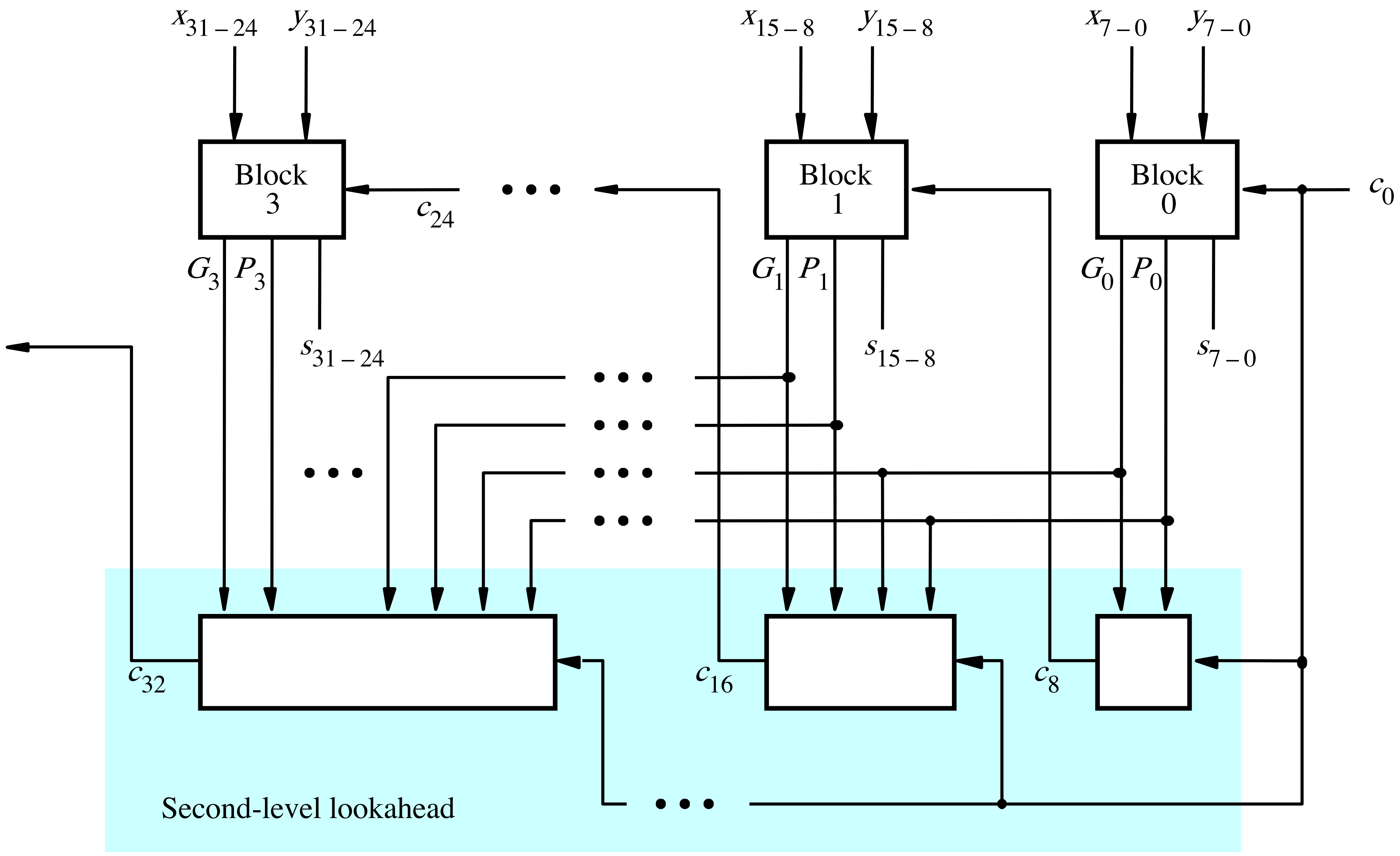
A hierarchical carry-lookahead adder

A hierarchical carry-lookahead adder with ripple-carry between blocks



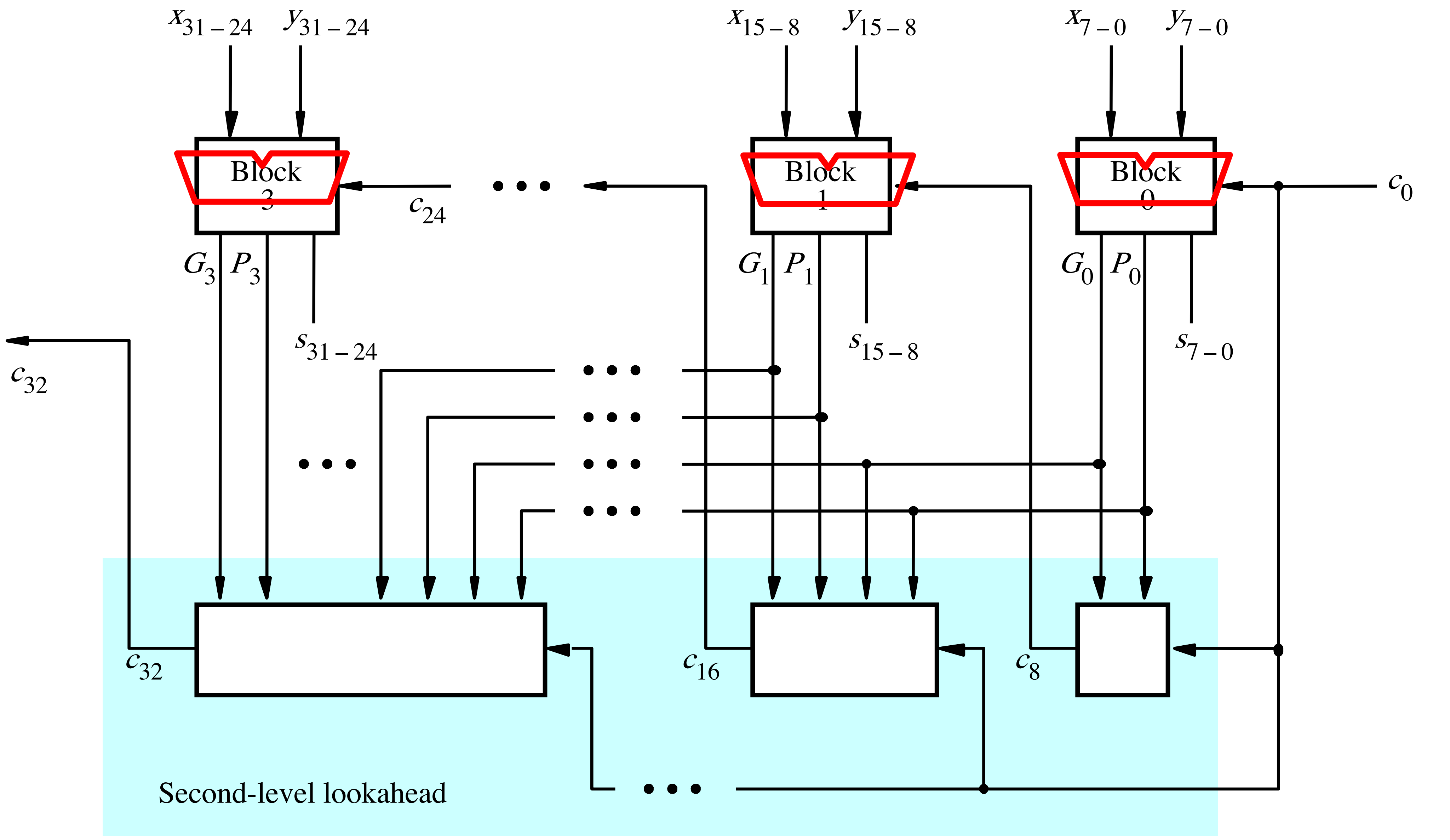
[Figure 3.16 from the textbook]

A hierarchical carry-lookahead adder

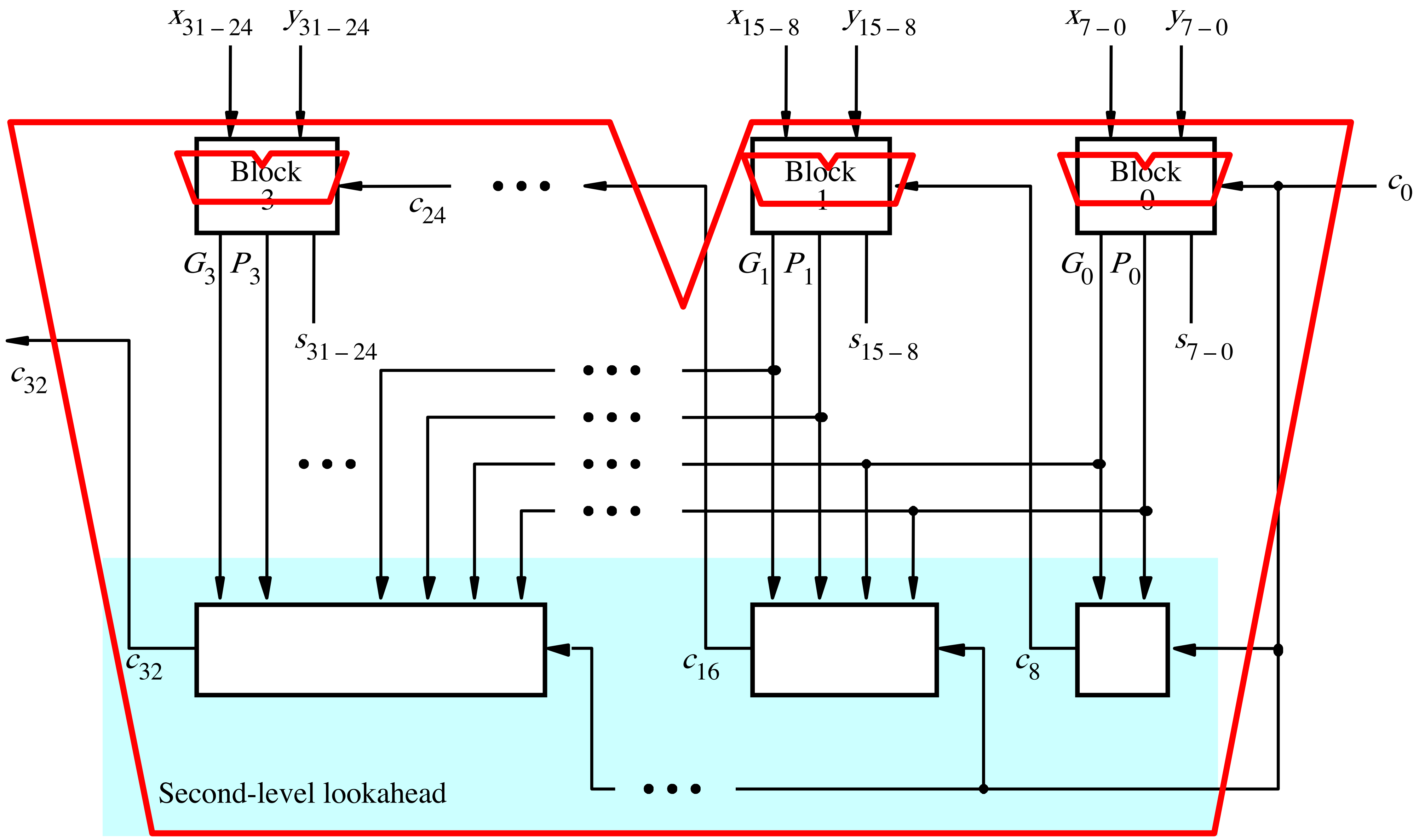


[Figure 3.17 from the textbook]

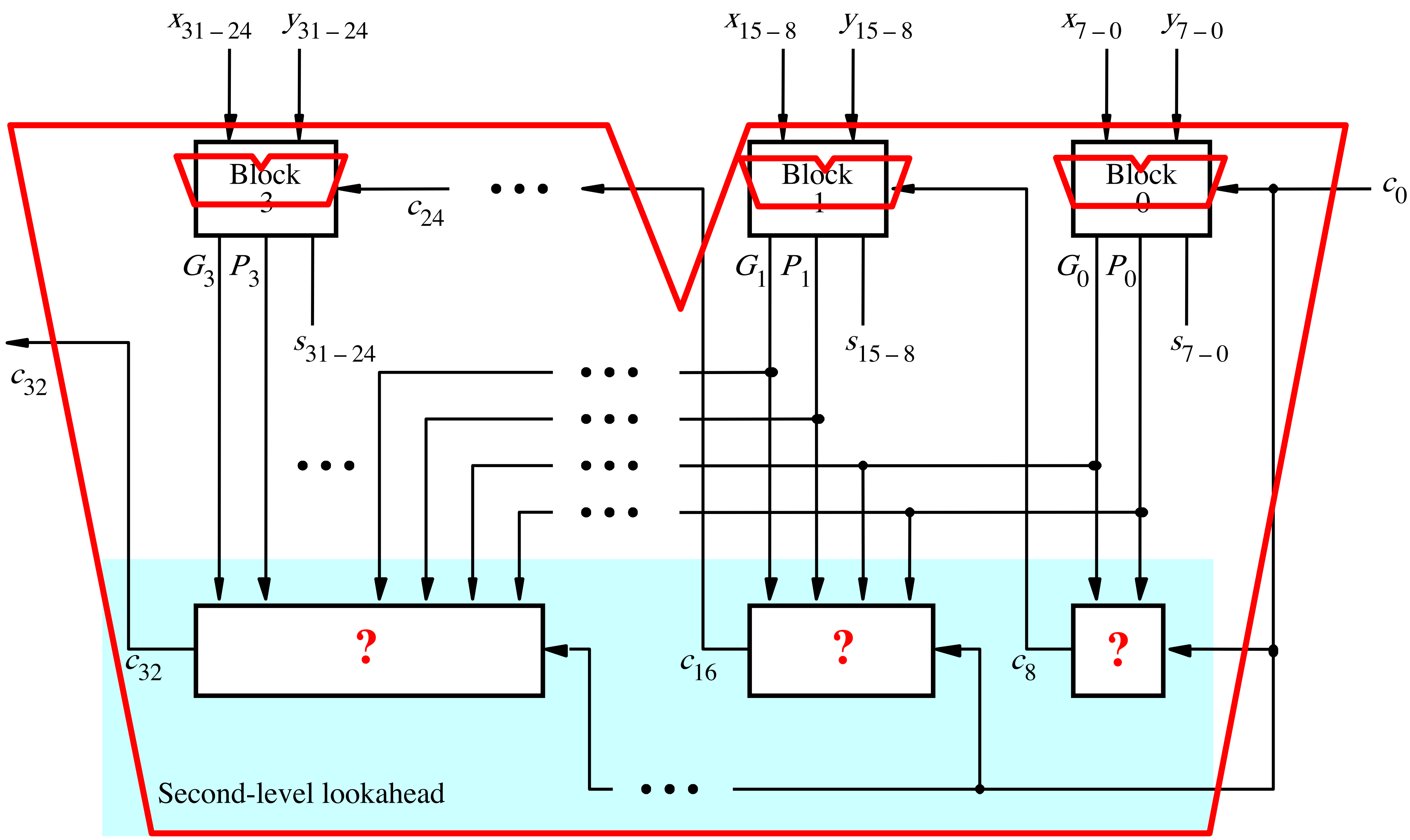
A hierarchical carry-lookahead adder



A hierarchical carry-lookahead adder



A hierarchical carry-lookahead adder



The Hierarchical Carry Expression

$$\begin{aligned} c_8 = \ & g_7 + p_7g_6 + p_7p_6g_5 + p_7p_6p_5g_4 \\ & + p_7p_6p_5p_4g_3 + p_7p_6p_5p_4p_3g_2 \\ & + p_7p_6p_5p_4p_3p_2g_1 + p_7p_6p_5p_4p_3p_2p_1g_0 \\ & + p_7p_6p_5p_4p_3p_2p_1p_0c_0 \end{aligned}$$

The Hierarchical Carry Expression

$$c_8 = \boxed{\begin{aligned} & g_7 + p_7g_6 + p_7p_6g_5 + p_7p_6p_5g_4 \\ &+ p_7p_6p_5p_4g_3 + p_7p_6p_5p_4p_3g_2 \\ &+ p_7p_6p_5p_4p_3p_2g_1 + p_7p_6p_5p_4p_3p_2p_1g_0 \end{aligned}}$$

$$+ \boxed{p_7p_6p_5p_4p_3p_2p_1p_0}\,c_0$$

The Hierarchical Carry Expression

$$c_8 = \underbrace{g_7 + p_7g_6 + p_7p_6g_5 + p_7p_6p_5g_4 + p_7p_6p_5p_4g_3 + p_7p_6p_5p_4p_3g_2 + p_7p_6p_5p_4p_3p_2g_1 + p_7p_6p_5p_4p_3p_2p_1g_0}_{G_0} + \underbrace{p_7p_6p_5p_4p_3p_2p_1p_0}_{P_0}c_0$$

The Hierarchical Carry Expression

$$c_8 = \underbrace{g_7 + p_7g_6 + p_7p_6g_5 + p_7p_6p_5g_4 + p_7p_6p_5p_4g_3 + p_7p_6p_5p_4p_3g_2 + p_7p_6p_5p_4p_3p_2g_1 + p_7p_6p_5p_4p_3p_2p_1g_0}_{\mathbf{G_0}} + \underbrace{p_7p_6p_5p_4p_3p_2p_1p_0}_{\mathbf{P_0}}c_0$$

$$c_8 = G_0 + P_0c_0$$

The Hierarchical Carry Expression

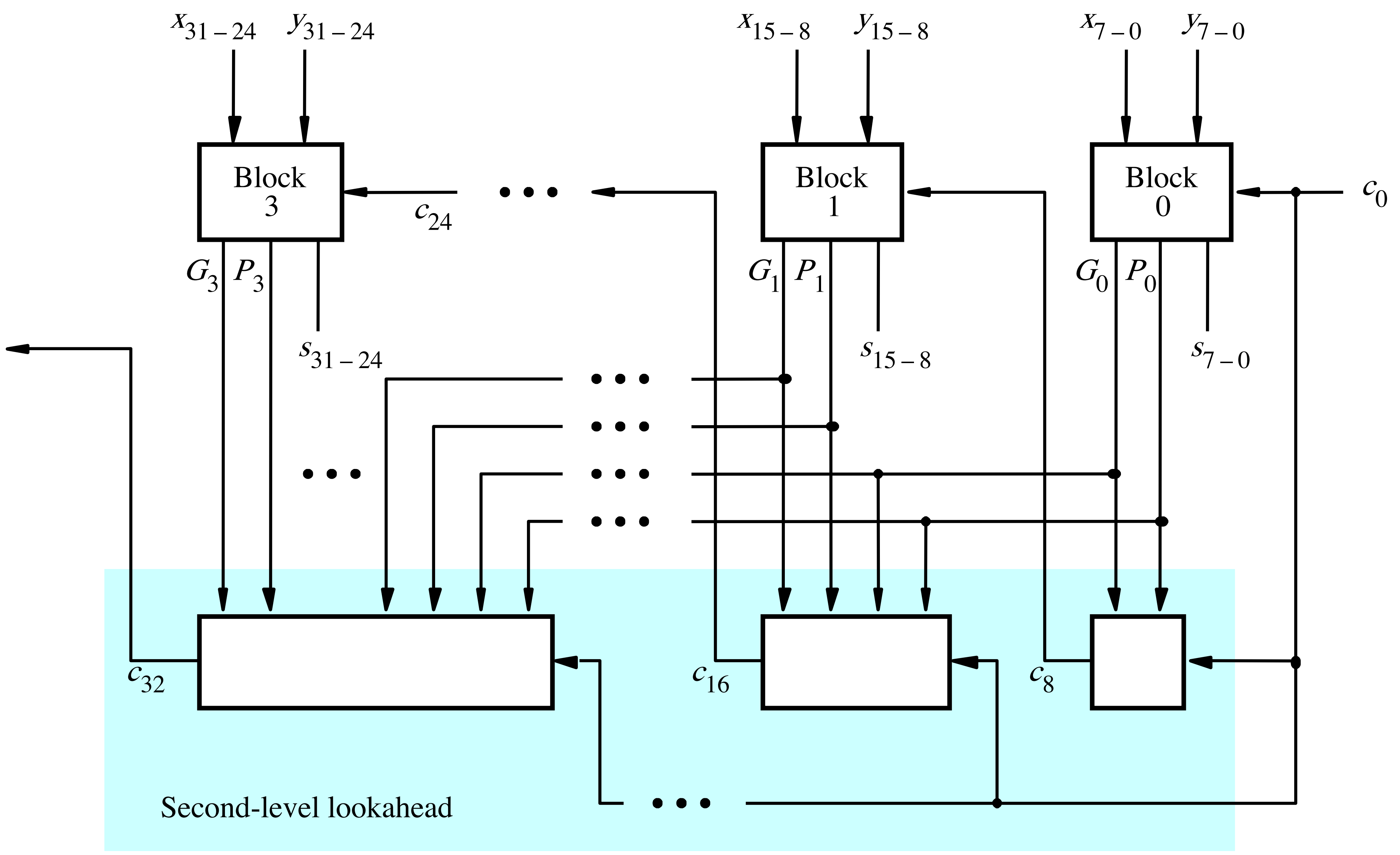
$$c_8 = G_0 + P_0 c_0$$

$$\begin{aligned} c_{16} &= G_1 + P_1 c_8 \\ &= G_1 + P_1 G_0 + P_1 P_0 c_0 \end{aligned}$$

$$c_{24} = G_2 + P_2 G_1 + P_2 P_1 G_0 + P_2 P_1 P_0 c_0$$

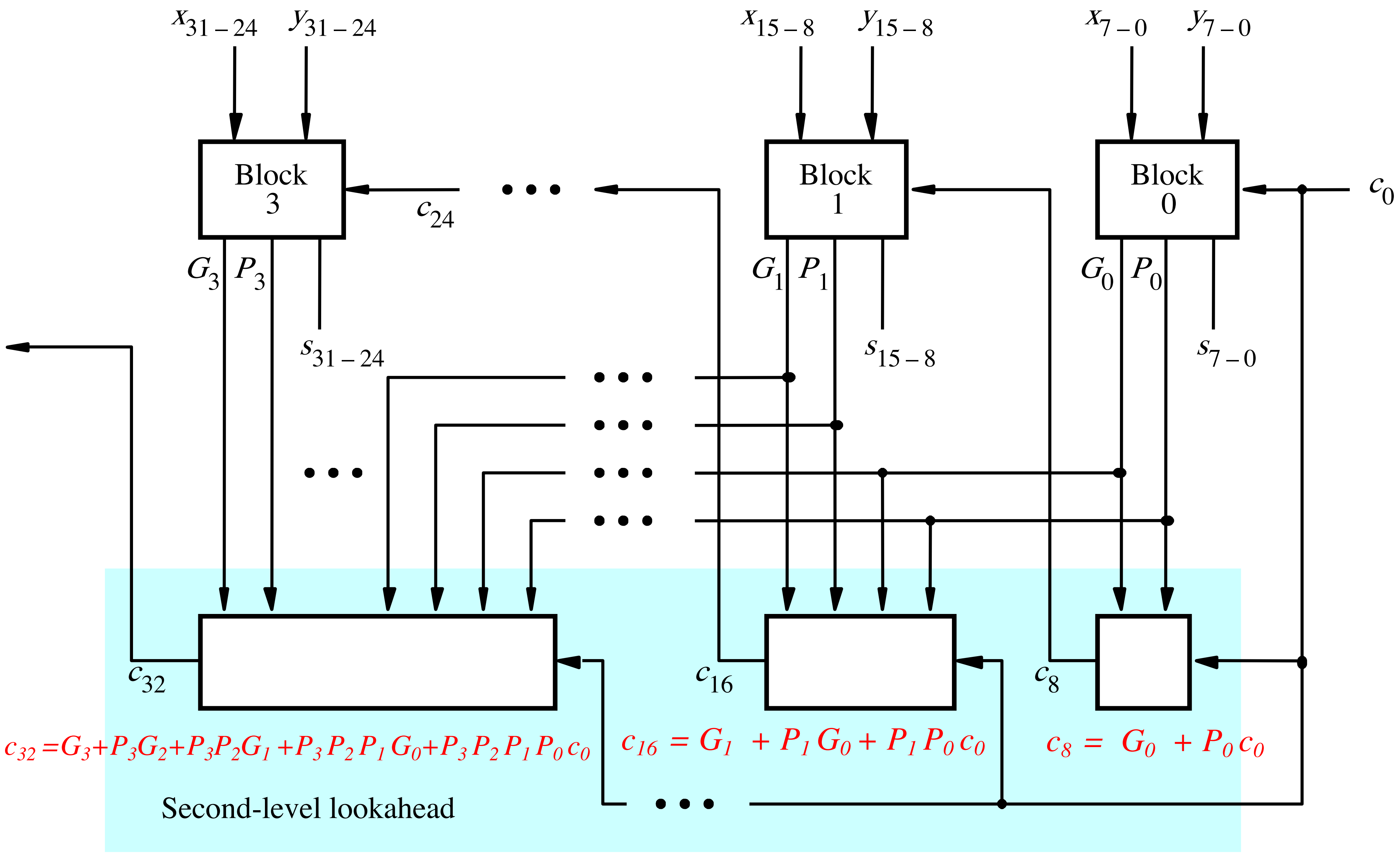
$$c_{32} = G_3 + P_3 G_2 + P_3 P_2 G_1 + P_3 P_2 P_1 G_0 + P_3 P_2 P_1 P_0 c_0$$

A hierarchical carry-lookahead adder



[Figure 3.17 from the textbook]

A hierarchical carry-lookahead adder



$c_{32} = G_3 + P_3G_2 + P_3P_2G_1 + P_3P_2P_1G_0 + P_3P_2P_1P_0c_0$

$c_{16} = G_1 + P_1G_0 + P_1P_0c_0$

$c_8 = G_0 + P_0c_0$

[Figure 3.17 from the textbook]

Total Gate Delay Through a Hierarchical Carry-Lookahead Adder

- **The total delay is 8 gates:**
  - **3 to generate all Gj and Pj signals**
  - **+2 to generate c8, c16, c24, and c32**
  - **+2 to generate internal carries in the blocks**
  - **+1 to generate the sum bits (one extra XOR)**

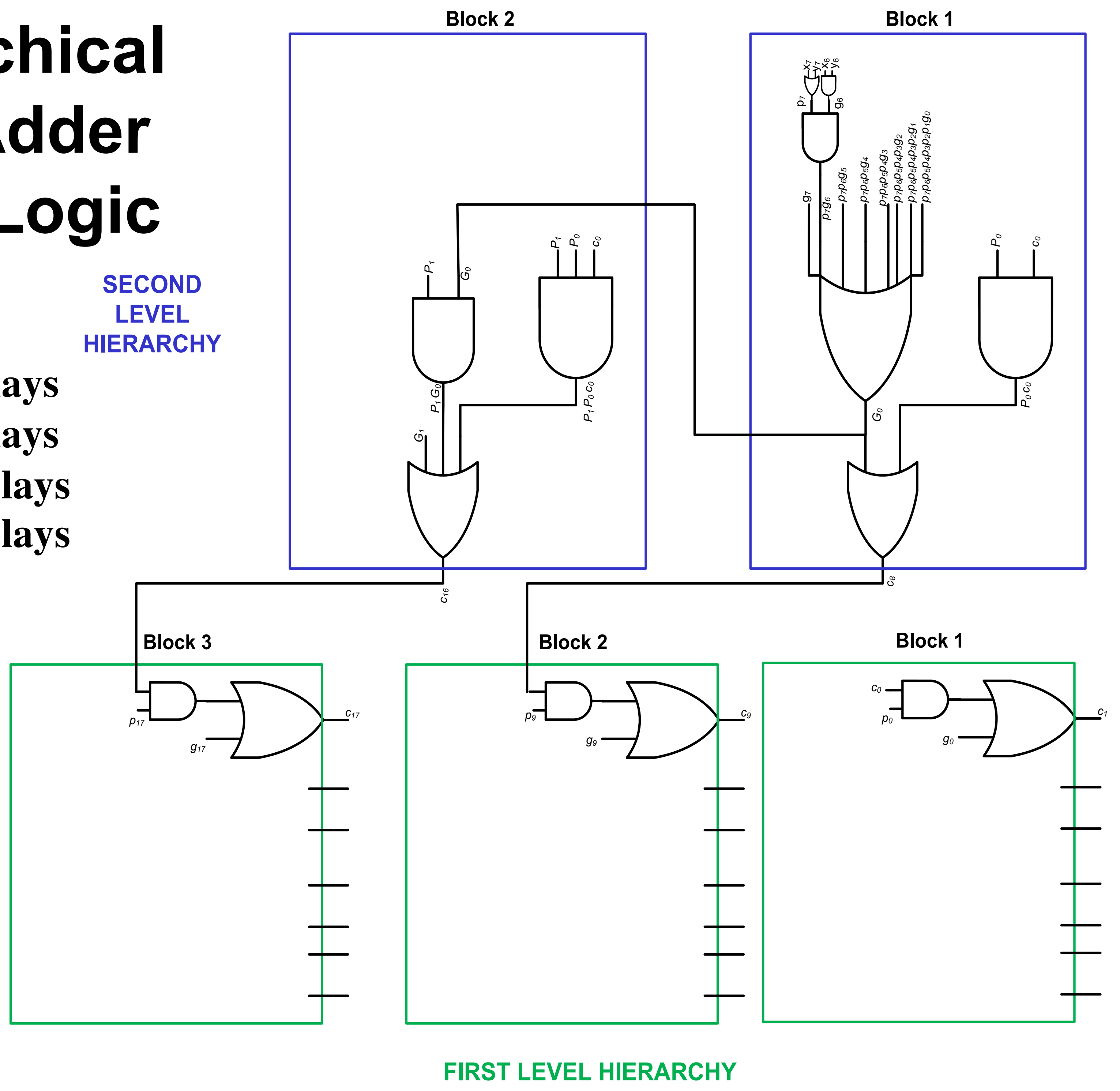
Hierarchical CLA Adder Carry Logic

C8 – 5 gate delays

C16 – 5 gate delays

C24 – 5 Gate delays

C32 – 5 Gate delays



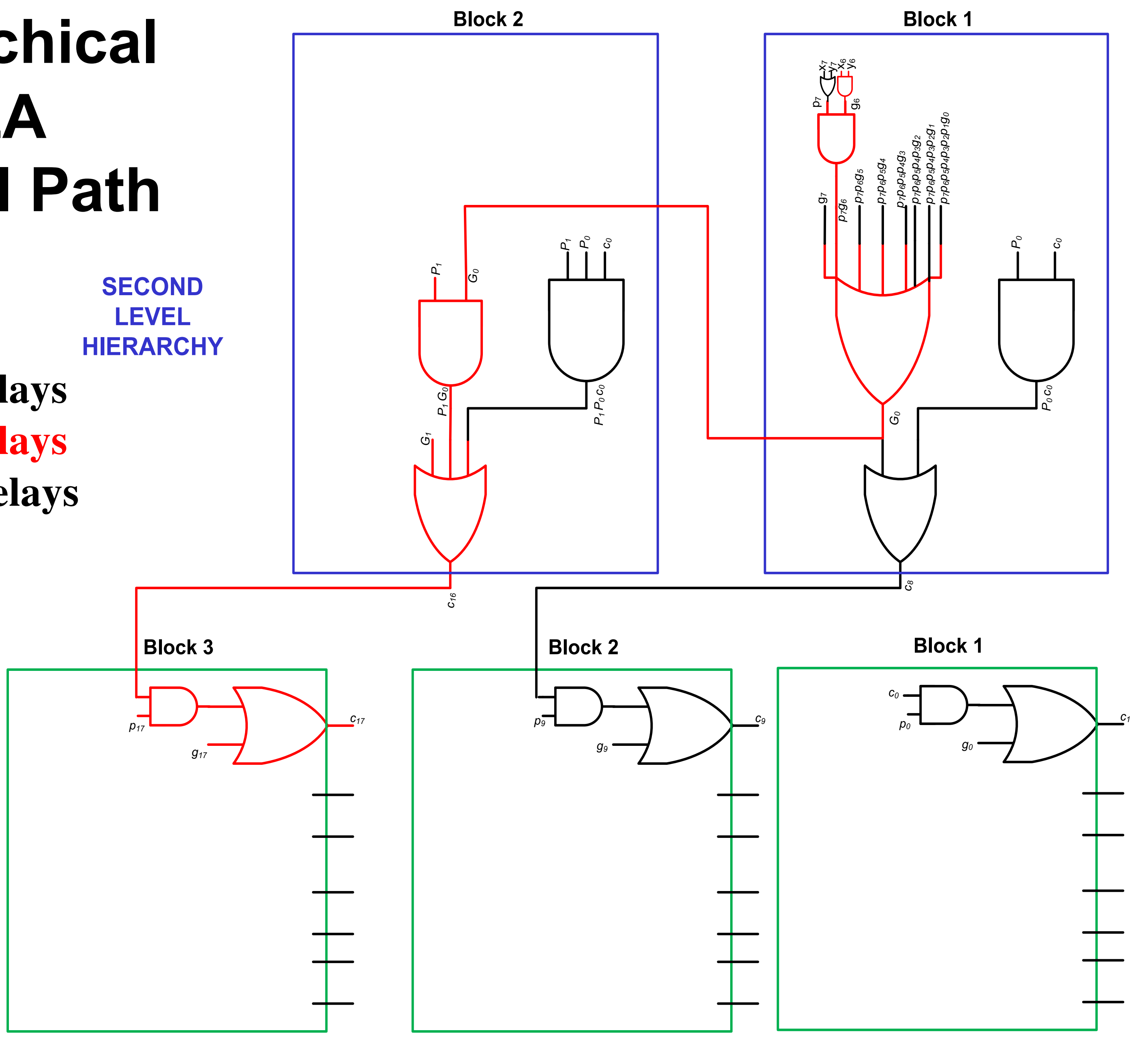
Hierarchical CLA Critical Path

SECOND LEVEL HIERARCHY

C9 – 7 gate delays

C17 – 7 gate delays

C25 – 7 Gate delays



FIRST LEVEL HIERARCHY

Questions?

THE END