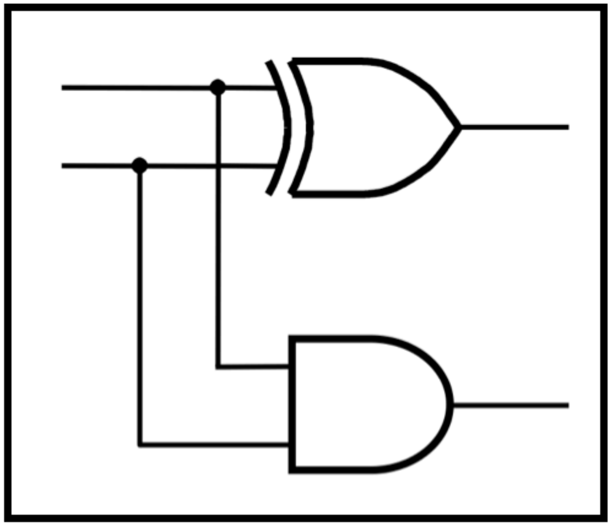


CprE 281: Digital Logic

Instructor: Alexander Stoytchev

http://www.ece.iastate.edu/~alexs/classes/

Synthesis Using AND, OR, and NOT Gates

CprE 281: Digital Logic
Iowa State University, Ames, IA

Administrative Stuff

- **HW2 is due on Wednesday Sep 8 @ 4pm**

- **Please write clearly on the first page the following three things:**

  - **Your First and Last Name**
  - **Your Student ID Number**
  - **Your Lab Section Letter**

- **Submit on Canvas as *one* PDF file.**

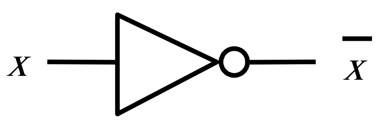
- **Please orient your pages such that the text can be read without the need to rotate the page.**

Administrative Stuff

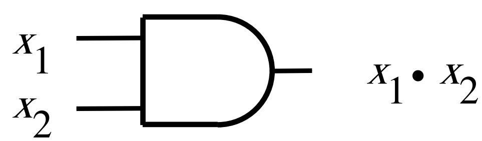
- Next week we will start with Lab2
- Read the lab assignment and do the prelab at home.
- Complete the prelab on paper before you go to the lab. Otherwise you'll lose 20% of your grade for that lab.

Quick Review

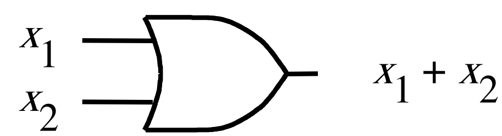
The Three Basic Logic Gates



NOT gate



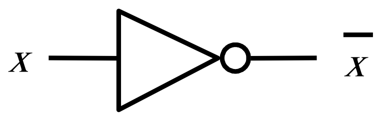
AND gate



OR gate

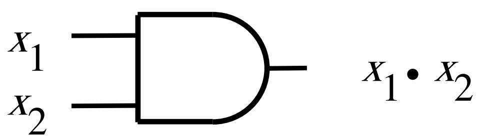
[Figure 2.8 from the textbook]

Truth Table for NOT



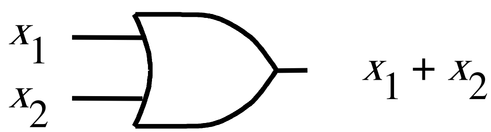
x	$\overline{x}$
0	1
1	0

Truth Table for AND



x_1	x_2	$x_1 \cdot x_2$
0	0	0
0	1	0
1	0	0
1	1	1

Truth Table for OR



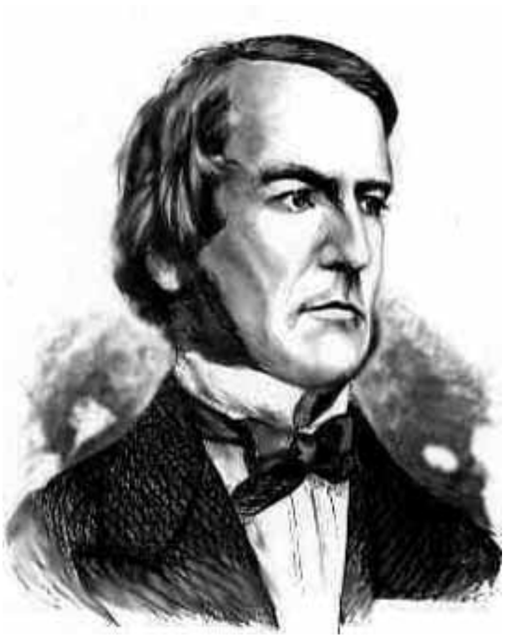
x_1	x_2	$x_1 + x_2$
0	0	0
0	1	1
1	0	1
1	1	1

Truth Tables for AND and OR

x_1	x_2	$x_1 \cdot x_2$	$x_1 + x_2$
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	1
		AND	OR

[Figure 2.6b from the textbook]

Boolean Algebra



George Boole
1815-1864

- **An algebraic structure consists of**
  - **a set of elements {0, 1}**
  - **binary operators {+, •}**
  - **and a unary operator { ' } or { ¯ } or { ~ }**
- **Introduced by George Boole in 1854**
- **An effective means of describing circuits built with switches**
- **A powerful tool that can be used for designing and analyzing logic circuits**

Different Notations for Negation

- **All three of these mean "negate x"**
  - x'
  - $\overline{x}$
  - $\sim x$

Operator Precedence

- In regular arithmetic and algebra, multiplication takes precedence over addition.
- This is also true in Boolean algebra.
- For example, $x + y \bullet z$ means
  multiply y by z and add the product to x .
- In other words, $x + y \bullet z$ is equal to $x + (y \bullet z)$, not $(x + y) \bullet z$.

The multiplication dot is optional

- In regular algebra, the multiplication operator is often omitted to shorten the equations.
- This is also true in Boolean algebra.
- Both of these mean the same thing:

  xy is equal to $x \cdot y$

Operator Precedence
(three different ways to write the same)

$$x_1 \cdot x_2 + \overline{x}_1 \cdot \overline{x}_2$$

$$(x_1 \cdot x_2) + ((\overline{x}_1) \cdot (\overline{x}_2))$$

$$x_1 x_2 + \overline{x}_1 \overline{x}_2$$

Operator Precedence

- **Negation of a single variable takes precedence over multiplication of that variable with another variable.**

- **For example,**

  $\overline{A}\,B$ means negate A first and then multiply $\overline{A}$ by B

Operator Precedence

- However, a horizontal bar over a product of two variables means that the negation is performed after the product is computed.

- For example,

  $\overline{A\,B}$ means multiply A and B and then negate

Operator Precedence

- **Note that these two expressions are different:**

$$\overline{A\,B} \text{ is not equal to } \overline{A}\,\overline{B}$$

$\overline{A\,B}$ **means multiply A and B and then negate**

$\overline{A}\,\overline{B}$ **means negate A and B separately and then multiply**

Operator Precedence

- **Note that these two expressions are different:**

$$\overline{A\,B} \text{ is not equal to } \overline{A}\,\overline{B}$$

A	B	$\overline{AB}$
0	0	1
0	1	1
1	0	1
1	1	0

A	B	$\overline{A}\overline{B}$
0	0	1
0	1	0
1	0	0
1	1	0

DeMorgan's Theorem

15a. $\overline{x \cdot y} = \overline{x} + \overline{y}$

15b. $\overline{x + y} = \overline{x} \cdot \overline{y}$

Proof of DeMorgan's theorem

15a. $\overline{x \cdot y} = \bar{x} + \bar{y}$

x	y	$x \cdot y$	$\overline{x \cdot y}$	$\bar{x}$	$\bar{y}$	$\bar{x} + \bar{y}$
0	0	0	1	1	1	1
0	1	0	1	1	0	1
1	0	0	1	0	1	1
1	1	1	0	0	0	0

LHS RHS

Proof of DeMorgan's theorem

15a. $\overline{x \cdot y} = \overline{x} + \overline{y}$

x	y	$x \cdot y$	$\overline{x \cdot y}$	$\overline{x}$	$\overline{y}$	$\overline{x} + \overline{y}$
0	0	0	1	1	1	1
0	1	0	1	1	0	1
1	0	0	1	0	1	1
1	1	1	0	0	0	0

LHS ($x \cdot y$, $\overline{x \cdot y}$) RHS ($\overline{x}$, $\overline{y}$, $\overline{x} + \overline{y}$)

These two columns are equal. Therefore, the theorem is true.

How to remember DeMorgan's theorem

$$\overline{x \cdot y}$$

start with the
left-hand side

How to remember DeMorgan’s theorem

$$\overline{x \cdot y}$$

divide the bar
into 3 equal parts

How to remember DeMorgan’s theorem

$$\overline{x} \bullet \overline{y}$$

erase the
middle segment

How to remember DeMorgan’s theorem

$$\overline{x} + \overline{y}$$

change the product to a sum

How to remember DeMorgan's theorem

$$\overline{x} + \overline{y}$$

this is the
right-hand side

How to remember DeMorgan's theorem

$$\overline{x \cdot y} = \overline{x} + \overline{y}$$

Proof of the other DeMorgan's theorem

15b. $\overline{x + y} = \overline{x} \bullet \overline{y}$

x	y	$x+y$	$\overline{x+y}$	$\overline{x}$	$\overline{y}$	$\overline{x} \bullet \overline{y}$
0	0	0	1	1	1	1
0	1	1	0	1	0	0
1	0	1	0	0	1	0
1	1	1	0	0	0	0

LHS RHS

Proof of the other DeMorgan's theorem

$$\textbf{15b.}\quad \overline{x + y} = \overline{x} \bullet \overline{y}$$

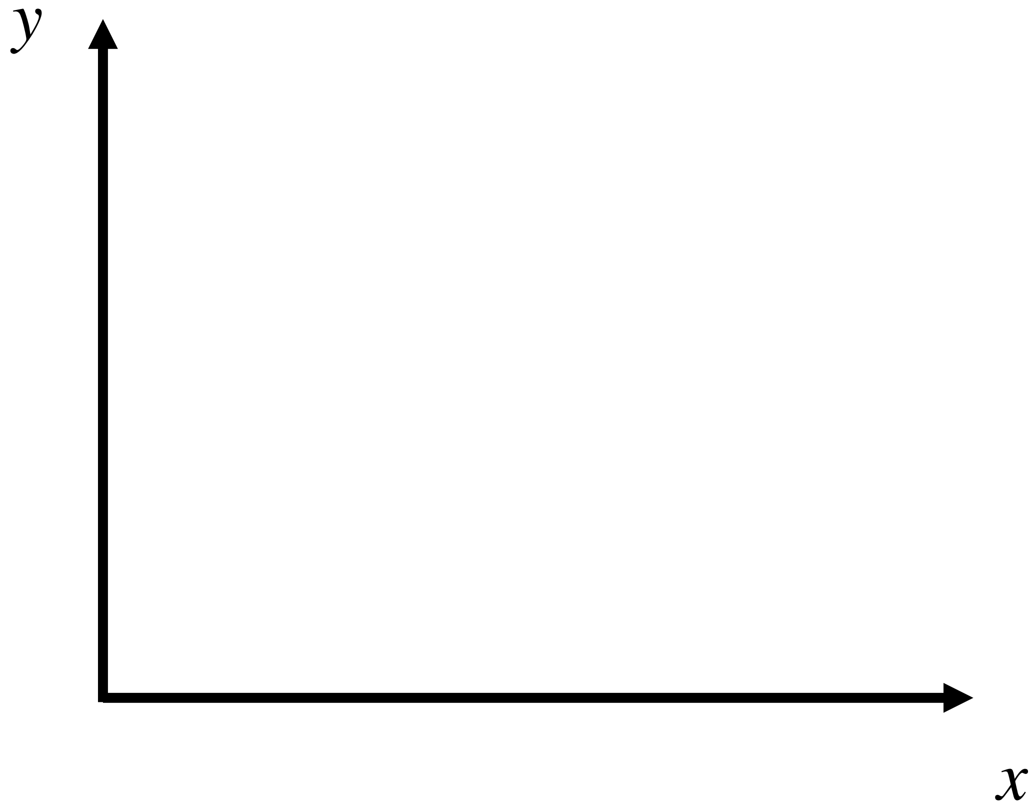
x	y	$x + y$	$\overline{x + y}$	$\overline{x}$	$\overline{y}$	$\overline{x} \bullet \overline{y}$
0	0	0	1	1	1	1
0	1	1	0	1	0	0
1	0	1	0	0	1	0
1	1	1	0	0	0	0

LHS (columns $x + y$, $\overline{x + y}$) RHS (columns $\overline{x}$, $\overline{y}$, $\overline{x} \bullet \overline{y}$)

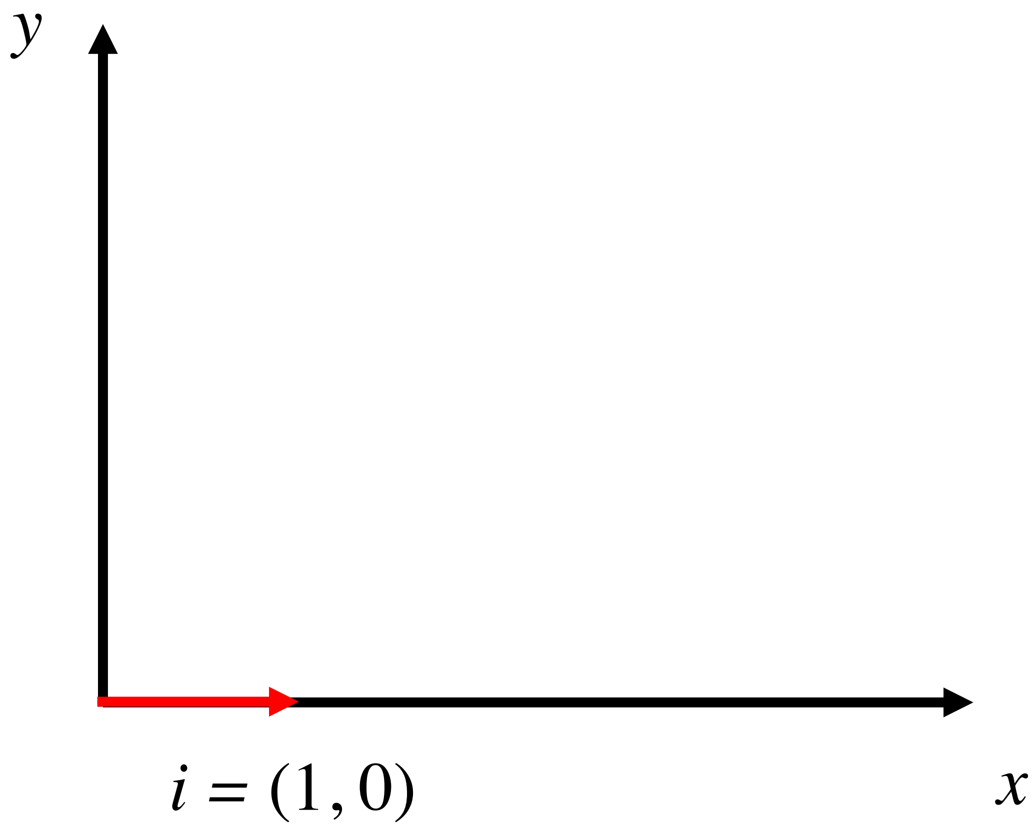
These two columns are equal. Therefore, the theorem is true.

A Short Digression

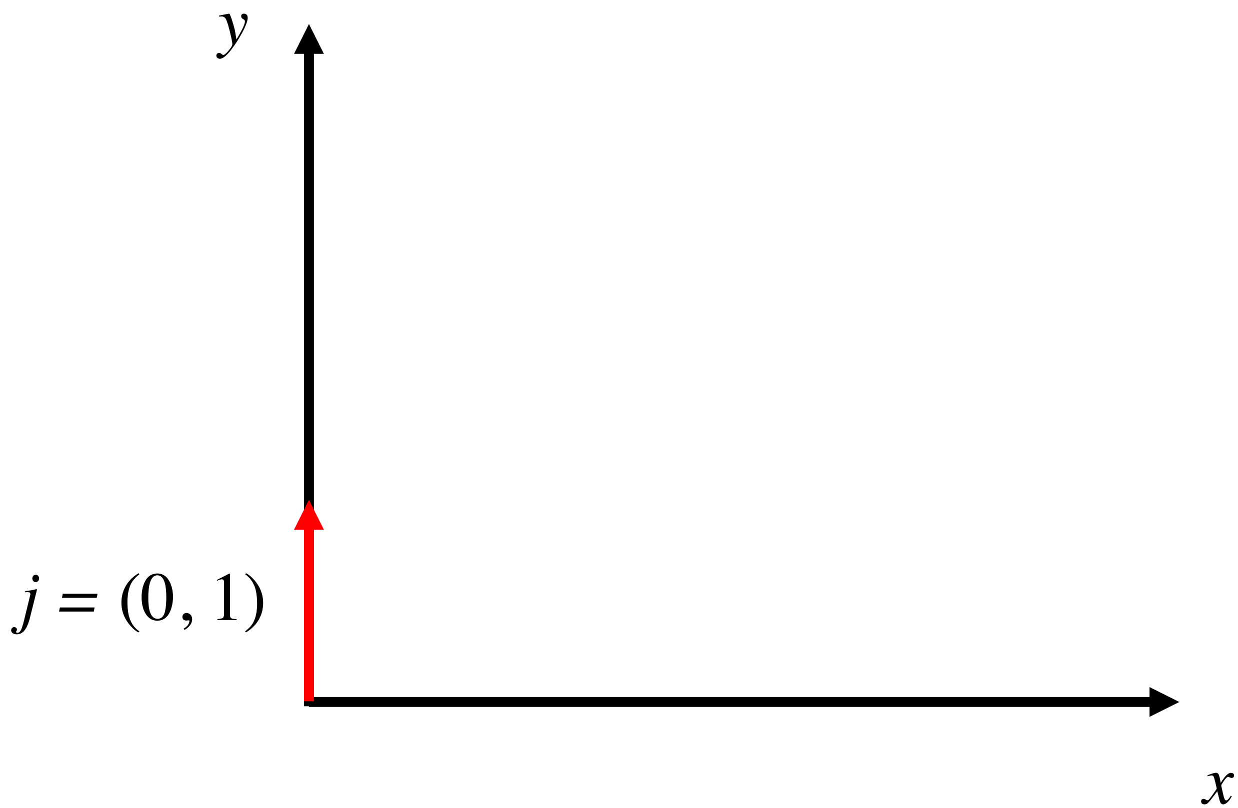
The 2D Plane



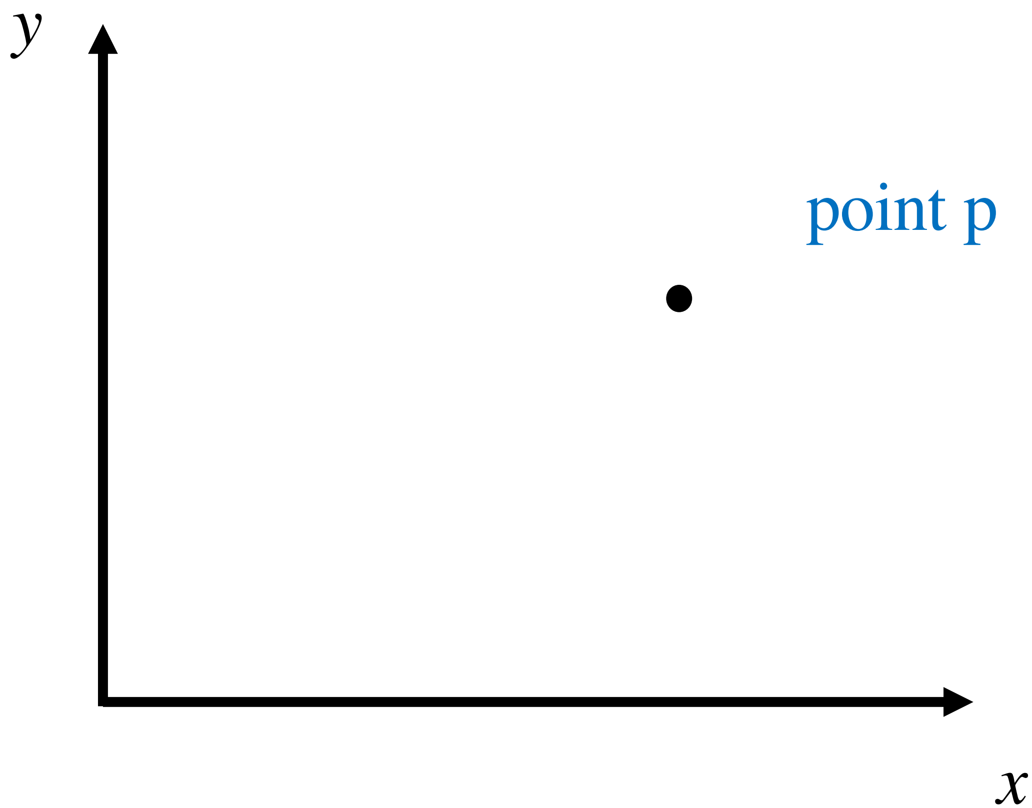
The 2D Plane



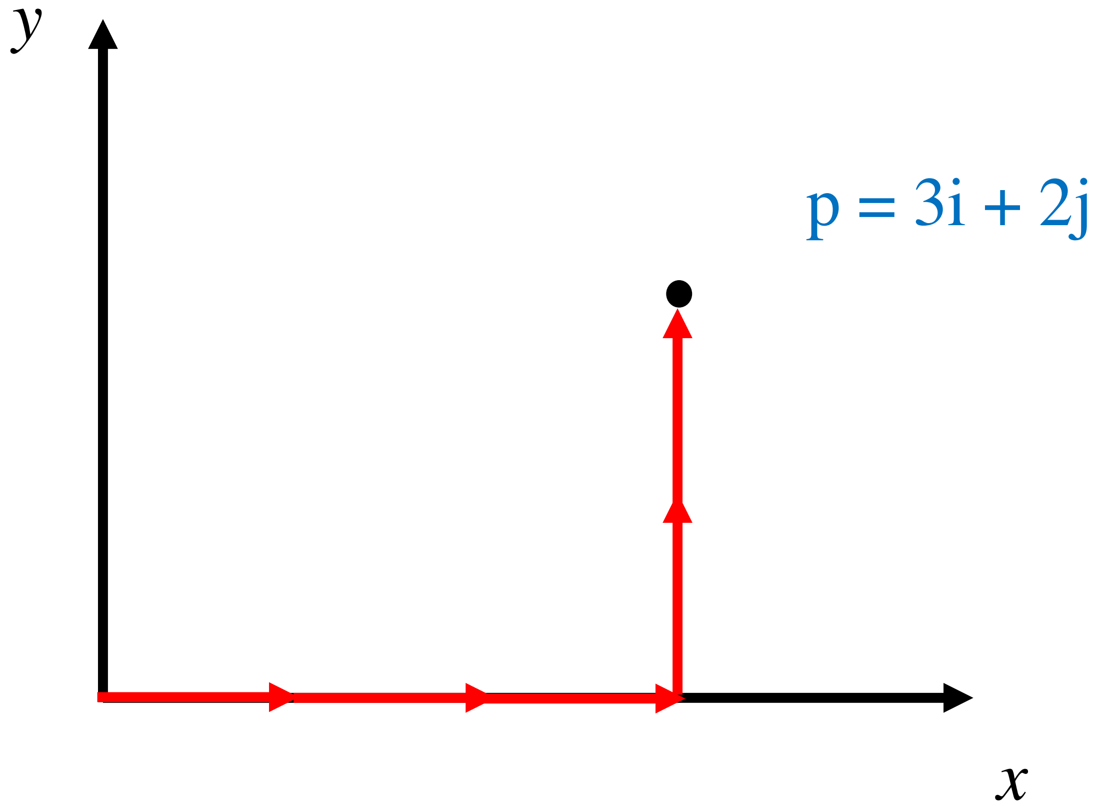
The 2D Plane



The 2D Plane



The 2D Plane



The unit vectors `i` and `j` form a basis

- **Any point in the 2D plane can be represented as a linear combination of these two vectors.**

- **In 3D we have `i, j,` and `k`**

```
i=(1, 0, 0)
j=(0, 1, 0)
k=(0, 0, 1)
```

Note that there is only one 1 in each.

Basis Functions
(for two variables)

Four Basis Functions

x	y	f_{00}
0	0	1
0	1	0
1	0	0
1	1	0

$f_{00}(x, y)$

x	y	f_{01}
0	0	0
0	1	1
1	0	0
1	1	0

$f_{01}(x, y)$

x	y	f_{10}
0	0	0
0	1	0
1	0	1
1	1	0

$f_{10}(x, y)$

x	y	f_{11}
0	0	0
0	1	0
1	0	0
1	1	1

$f_{11}(x, y)$

Four Basis Functions

x	y	f_{00}
0	0	1
0	1	0
1	0	0
1	1	0

$f_{00}(x, y)$

x	y	f_{01}
0	0	0
0	1	1
1	0	0
1	1	0

$f_{01}(x, y)$

x	y	f_{10}
0	0	0
0	1	0
1	0	1
1	1	0

$f_{10}(x, y)$

x	y	f_{11}
0	0	0
0	1	0
1	0	0
1	1	1

$f_{11}(x, y)$

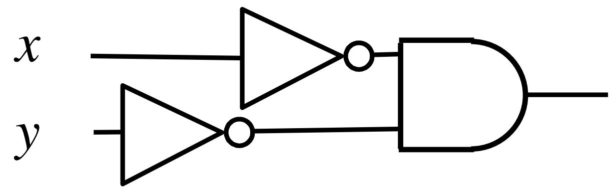
Four Basis Functions

x	y		$f_{00}(x, y)$	$f_{01}(x, y)$	$f_{10}(x, y)$	$f_{11}(x, y)$
0	0		$\boxed{1}$	0	0	0
0	1		0	$\boxed{1}$	0	0
1	0		0	0	$\boxed{1}$	0
1	1		0	0	0	$\boxed{1}$

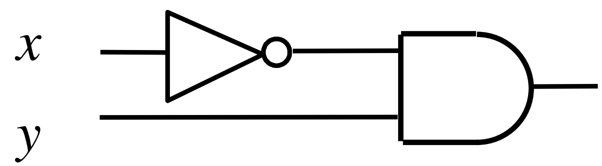
Four Basis Functions

x	y		$\bar{x}\,\bar{y}$	$\bar{x}\,y$	$x\,\bar{y}$	$x\,y$
0	0		1	0	0	0
0	1		0	1	0	0
1	0		0	0	1	0
1	1		0	0	0	1

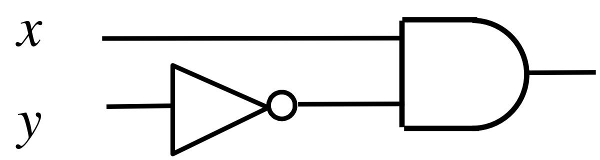
Circuits for the four basis functions



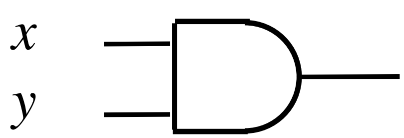
$$f_{00}(x, y) = \overline{x}\;\overline{y}$$



$$f_{01}(x, y) = \overline{x}\; y$$



$$f_{10}(x, y) = x\; \overline{y}$$



$$f_{11}(x, y) = x\; y$$

Four Basis Functions

x	y	f_{00}
0	0	1
0	1	0
1	0	0
1	1	0

x	y	f_{01}
0	0	0
0	1	1
1	0	0
1	1	0

x	y	f_{10}
0	0	0
0	1	0
1	0	1
1	1	0

x	y	f_{11}
0	0	0
0	1	0
1	0	0
1	1	1

$f_{00}(x, y) = \overline{x}\,\overline{y}$ $f_{01}(x, y) = \overline{x}\,y$ $f_{10}(x, y) = x\,\overline{y}$ $f_{11}(x, y) = x\,y$

Function Synthesis Example
(with two variables)

Synthesize the Following Function

x_1	x_2	$f(x_1, x_2)$
0	0	1
0	1	1
1	0	0
1	1	1

1) Split the function into a sum of 4 functions

x_1	x_2	$f(x_1, x_2)$		$f_{00}(x_1, x_2)$	$f_{01}(x_1, x_2)$	$f_{10}(x_1, x_2)$	$f_{11}(x_1, x_2)$
0	0	1		1	0	0	0
0	1	1		0	1	0	0
1	0	0		0	0	1	0
1	1	1		0	0	0	1

1) Split the function into a sum of 4 functions

x_1	x_2	$f(x_1, x_2)$		$f_{00}(x_1, x_2)$	$f_{01}(x_1, x_2)$	$f_{10}(x_1, x_2)$	$f_{11}(x_1, x_2)$
0	0	1		1	0	0	0
0	1	1		0	1	0	0
1	0	0		0	0	1	0
1	1	1		0	0	0	1

$$f(x_1, x_2) = 1 \bullet f_{00} \quad + \quad 1 \bullet f_{01} \quad + \quad 0 \bullet f_{10} \quad + \quad 1 \bullet f_{11}$$

2) Write the expressions for all four

x_1	x_2	$f(x_1, x_2)$		$f_{00}(x_1, x_2)$	$f_{01}(x_1, x_2)$	$f_{10}(x_1, x_2)$	$f_{11}(x_1, x_2)$
0	0	1		1	0	0	0
0	1	1		0	1	0	0
1	0	0		0	0	1	0
1	1	1		0	0	0	1

$$f(x_1, x_2) = 1 \bullet f_{00} + 1 \bullet f_{01} + 0 \bullet f_{10} + 1 \bullet f_{11}$$

2) Write the expressions for all four

x_1	x_2	$f(x_1, x_2)$		$f_{00}(x_1, x_2)$	$f_{01}(x_1, x_2)$	$f_{10}(x_1, x_2)$	$f_{11}(x_1, x_2)$
0	0	1		1	0	0	0
0	1	1		0	1	0	0
1	0	0		0	0	1	0
1	1	1		0	0	0	1

$$f(x_1, x_2) = \underbrace{1 \bullet f_{00}}_{\bar{x}_1\bar{x}_2} + \underbrace{1 \bullet f_{01}}_{\bar{x}_1 x_2} + \underbrace{0 \bullet f_{10}}_{0} + \underbrace{1 \bullet f_{11}}_{x_1 x_2}$$

3) Then just add them together

x_1	x_2	$f(x_1, x_2)$		$f_{00}(x_1, x_2)$	$f_{01}(x_1, x_2)$	$f_{10}(x_1, x_2)$	$f_{11}(x_1, x_2)$
0	0	1		1	0	0	0
0	1	1		0	1	0	0
1	0	0		0	0	1	0
1	1	1		0	0	0	1

$$f(x_1, x_2) = \underbrace{1 \bullet f_{00}} + \underbrace{1 \bullet f_{01}} + \underbrace{0 \bullet f_{10}} + \underbrace{1 \bullet f_{11}}$$

$$f(x_1, x_2) = \bar{x}_1\bar{x}_2 + \bar{x}_1 x_2 + 0 + x_1 x_2$$

3) Then just add them together

x_1	x_2	$f(x_1, x_2)$		$f_{00}(x_1, x_2)$	$f_{01}(x_1, x_2)$	$f_{10}(x_1, x_2)$	$f_{11}(x_1, x_2)$
0	0	1		$\boxed{1}$	0	0	0
0	1	1		0	$\boxed{1}$	0	0
1	0	0		0	0	1	0
1	1	1		0	0	0	$\boxed{1}$

$$f(x_1, x_2) = \bar{x}_1\bar{x}_2 + \bar{x}_1 x_2 + 0 + x_1 x_2$$

A function to be synthesized

x_1	x_2	$f(x_1, x_2)$
0	0	1
0	1	1
1	0	0
1	1	1

[Figure 2.19 from the textbook]

Let's look at it row by row.
How can we express the last row?

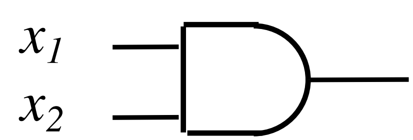
x_1	x_2	$f(x_1, x_2)$
0	0	1
0	1	1
1	0	0
1	1	1

Let’s look at it row by row.
How can we express the last row?

x_1	x_2	$f(x_1, x_2)$	
0	0	1	
0	1	1	
1	0	0	
1	1	$\boxed{1}$	x_1x_2

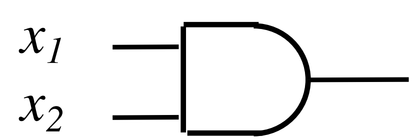
Let's look at it row by row.
How can we express the last row?

x_1	x_2	$f(x_1, x_2)$
0	0	1
0	1	1
1	0	0
1	1	$\boxed{1}$



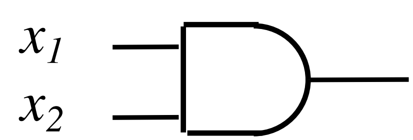
What about this row?

x_1	x_2	$f(x_1, x_2)$
0	0	1
0	1	1
1	0	0
1	1	1



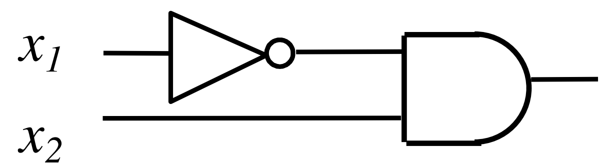
What about this row?

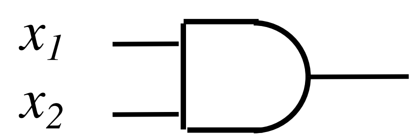
x_1	x_2	$f(x_1, x_2)$	
0	0	1	
0	1	$\boxed{1}$	$\bar{x}_1 x_2$
1	0	0	
1	1	1	



What about this row?

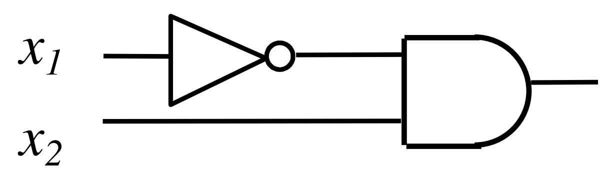
x_1	x_2	$f(x_1, x_2)$
0	0	1
0	1	$\boxed{1}$
1	0	0
1	1	1

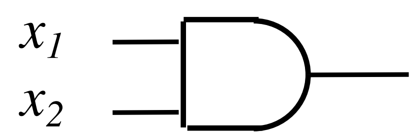




What about the first row?

x_1	x_2	$f(x_1, x_2)$
0	0	1
0	1	1
1	0	0
1	1	1

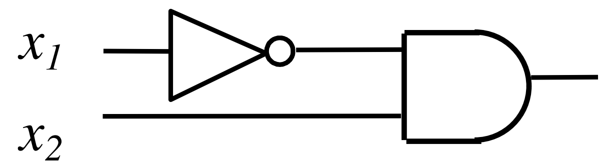


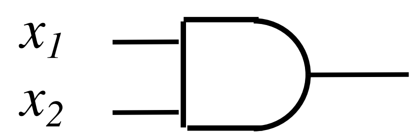


What about the first row?

x_1	x_2	$f(x_1, x_2)$
0	0	1
0	1	1
1	0	0
1	1	1

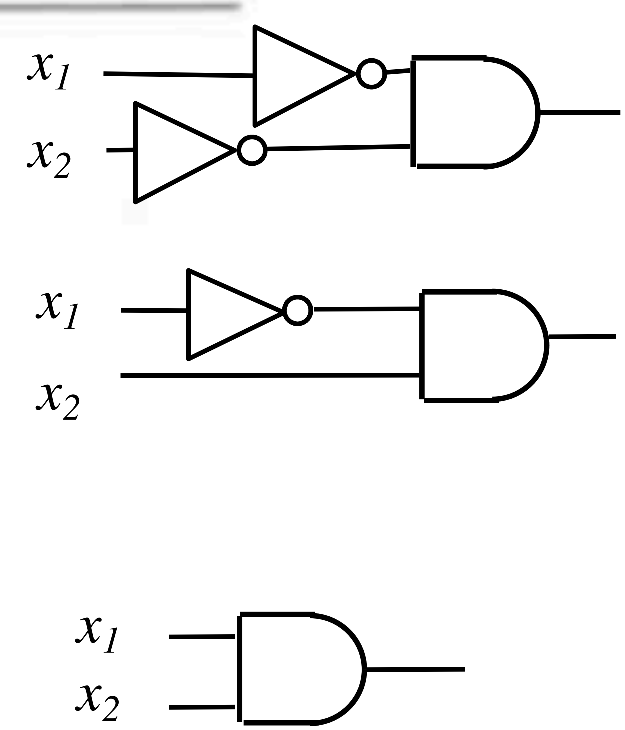
$\bar{x}_1\bar{x}_2$





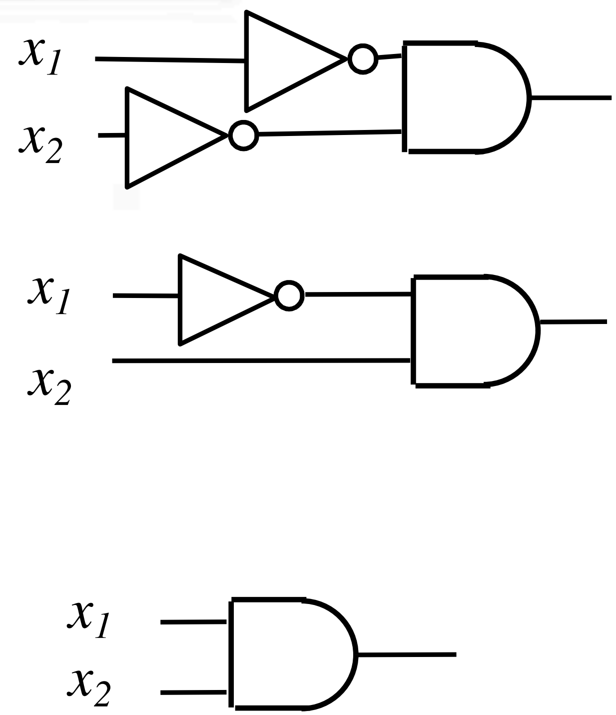
What about the first row?

x_1	x_2	$f(x_1, x_2)$
0	0	1
0	1	1
1	0	0
1	1	1

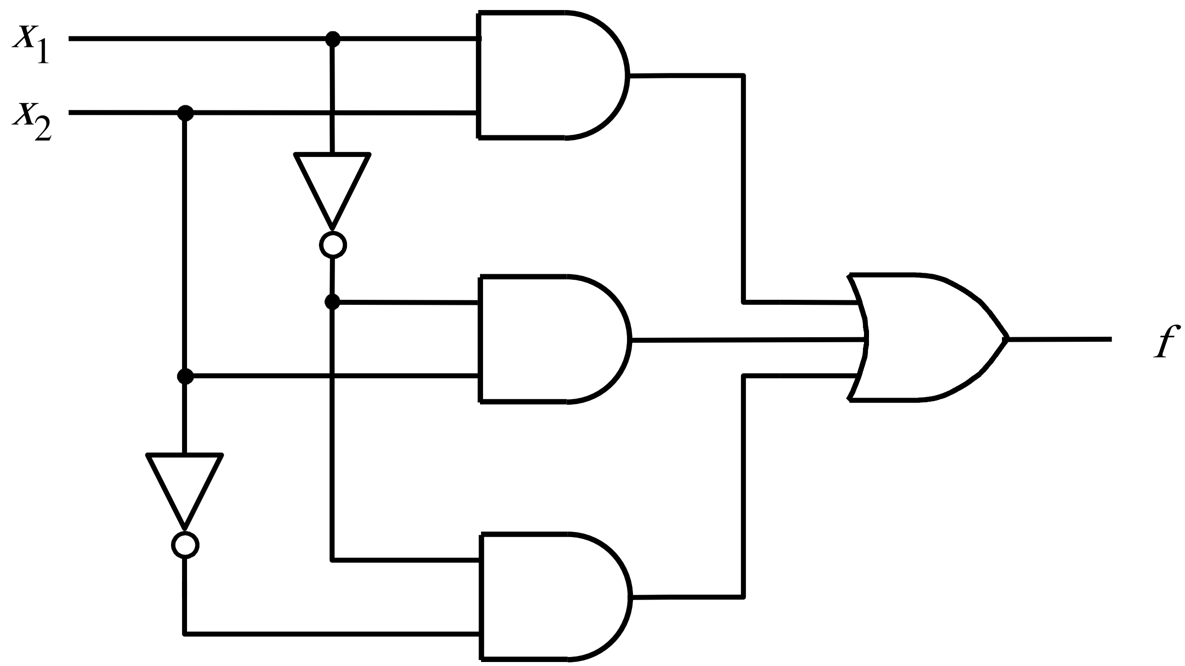


Finally, what about the zero?

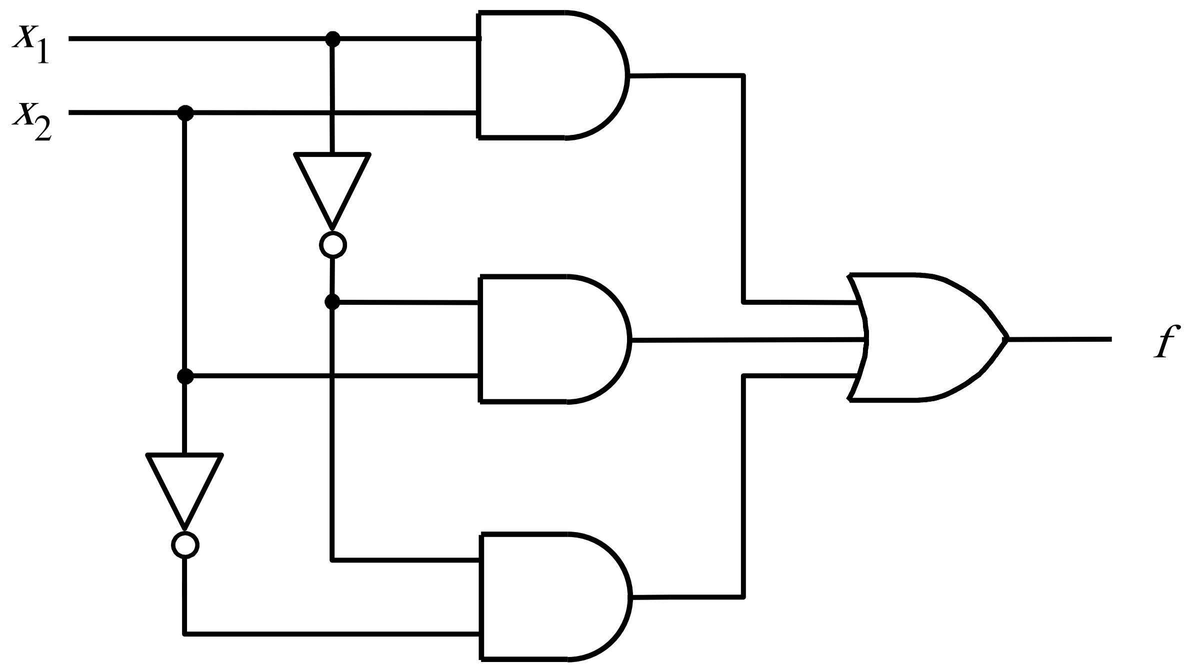
x_1	x_2	$f(x_1, x_2)$
0	0	1
0	1	1
1	0	0
1	1	1



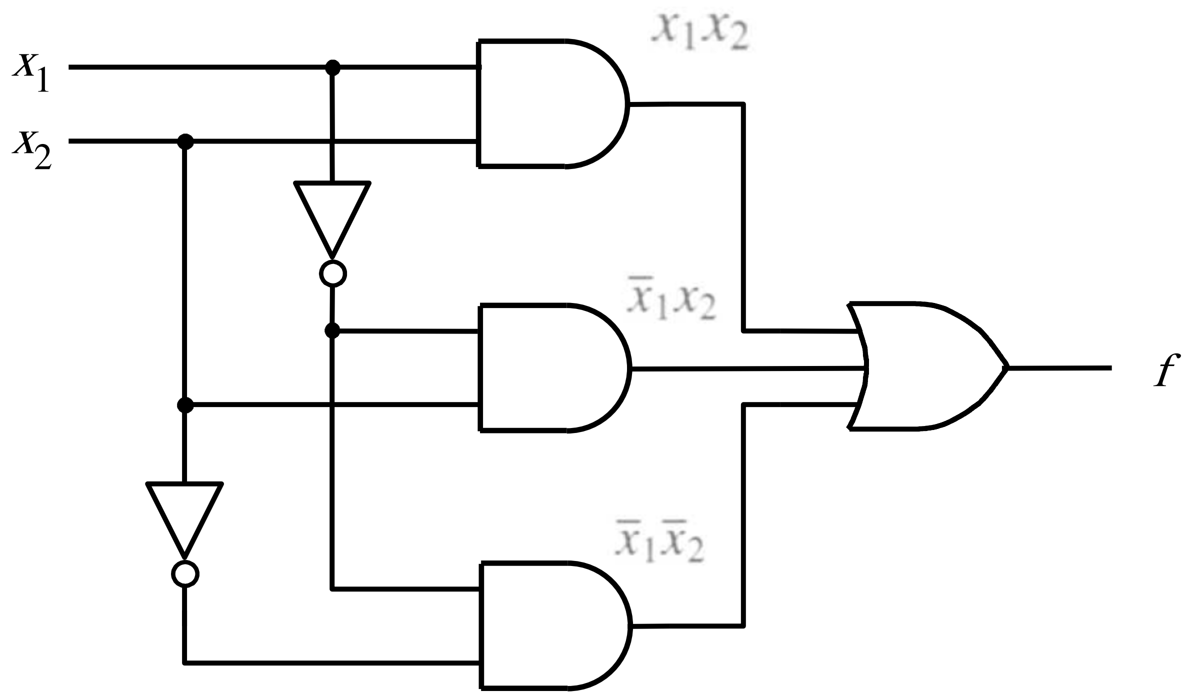
Putting it all together



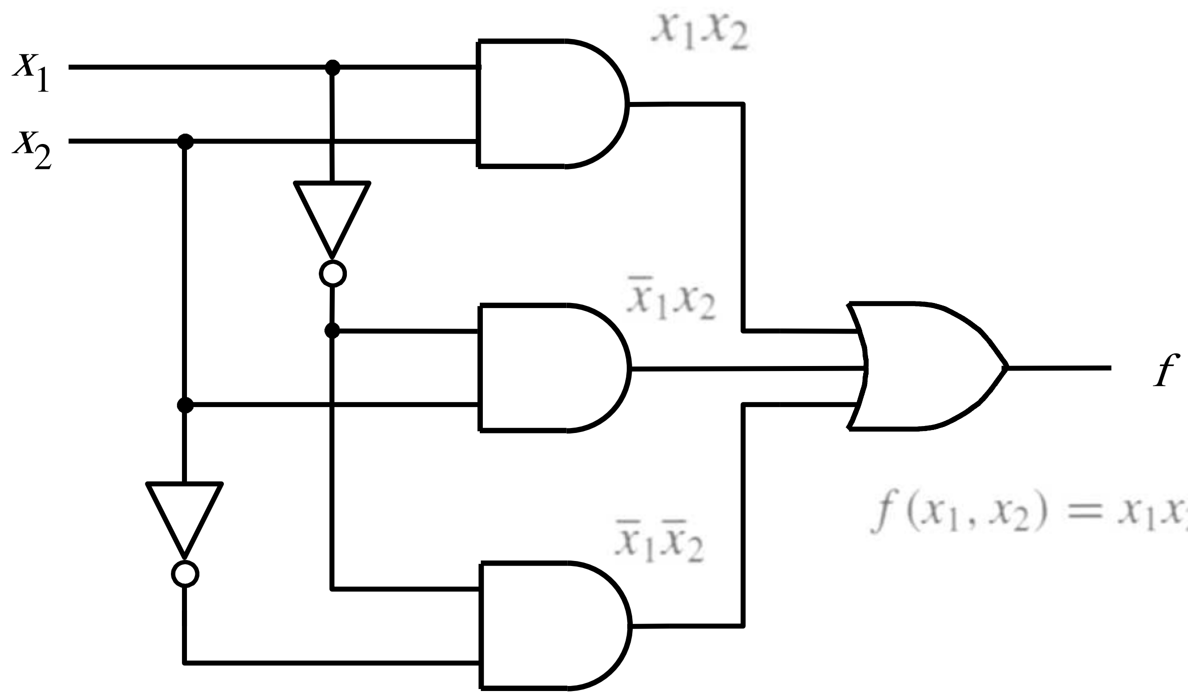
Let's verify that this circuit implements correctly the target truth table



Let's verify that this circuit implements correctly the target truth table

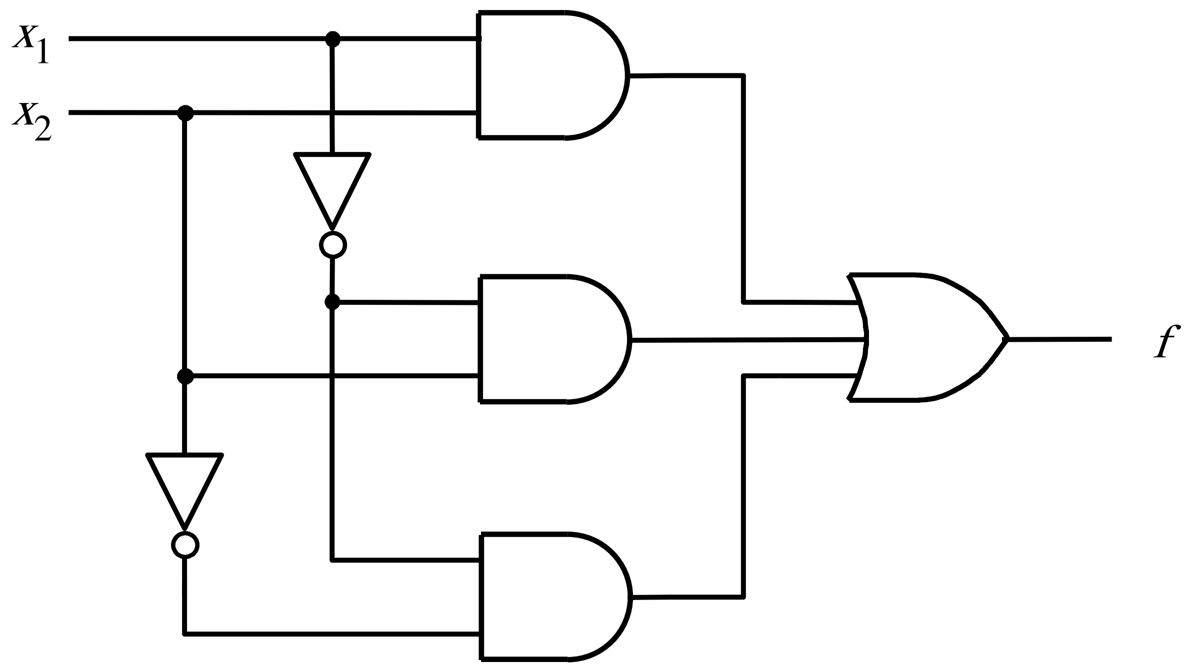


Putting it all together



$$f(x_1, x_2) = x_1x_2 + \bar{x}_1\bar{x}_2 + \bar{x}_1x_2$$

Canonical Sum-Of-Products (SOP)



$$f(x_1, x_2) = x_1x_2 + \bar{x}_1\bar{x}_2 + \bar{x}_1x_2$$

[Figure 2.20a from the textbook]

Summary of This Procedure

- **Get the truth table of the function**
- **Form a product term (AND gate) for each row of the table for which the function is 1**
- **Each product term contains all input variables**
- **In each row, if x_i=1 enter it as x_i, otherwise use $\overline{x}_i$**
- **Sum all of these products (OR gate) to get the function**

Simplification Steps

$$f(x_1, x_2) = x_1x_2 + \bar{x}_1\bar{x}_2 + \bar{x}_1x_2$$

Simplification Steps

$$f(x_1, x_2) = x_1 x_2 + \bar{x}_1 \bar{x}_2 + \boxed{\bar{x}_1 x_2}$$

replicate
this term

$$f(x_1, x_2) = x_1 x_2 + \bar{x}_1 \bar{x}_2 + \bar{x}_1 x_2 + \bar{x}_1 x_2$$

Simplification Steps

$$f(x_1, x_2) = x_1x_2 + \bar{x}_1\bar{x}_2 + \bar{x}_1x_2$$

group
these terms

$$f(x_1, x_2) = \boxed{x_1x_2} + \boxed{\bar{x}_1\bar{x}_2} + \boxed{\bar{x}_1x_2} + \boxed{\bar{x}_1x_2}$$

$$f(x_1, x_2) = (x_1 + \bar{x}_1)x_2 + \bar{x}_1(\bar{x}_2 + x_2)$$

Simplification Steps

$$f(x_1, x_2) = x_1 x_2 + \bar{x}_1 \bar{x}_2 + \bar{x}_1 x_2$$

$$f(x_1, x_2) = x_1 x_2 + \bar{x}_1 \bar{x}_2 + \bar{x}_1 x_2 + \bar{x}_1 x_2$$

These two terms are trivially equal to 1

$$f(x_1, x_2) = (\boxed{x_1 + \bar{x}_1}) x_2 + \bar{x}_1 (\boxed{\bar{x}_2 + x_2})$$

$$f(x_1, x_2) = 1 \cdot x_2 + \bar{x}_1 \cdot 1$$

Simplification Steps

$$f(x_1, x_2) = x_1x_2 + \bar{x}_1\bar{x}_2 + \bar{x}_1x_2$$

$$f(x_1, x_2) = x_1x_2 + \bar{x}_1\bar{x}_2 + \bar{x}_1x_2 + \bar{x}_1x_2$$

$$f(x_1, x_2) = (x_1 + \bar{x}_1)x_2 + \bar{x}_1(\bar{x}_2 + x_2)$$

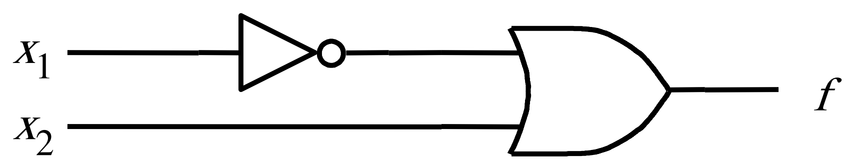
Drop the 1's

$$f(x_1, x_2) = \boxed{1} \cdot x_2 + \bar{x}_1 \cdot \boxed{1}$$

$$f(x_1, x_2) = x_2 + \bar{x}_1$$

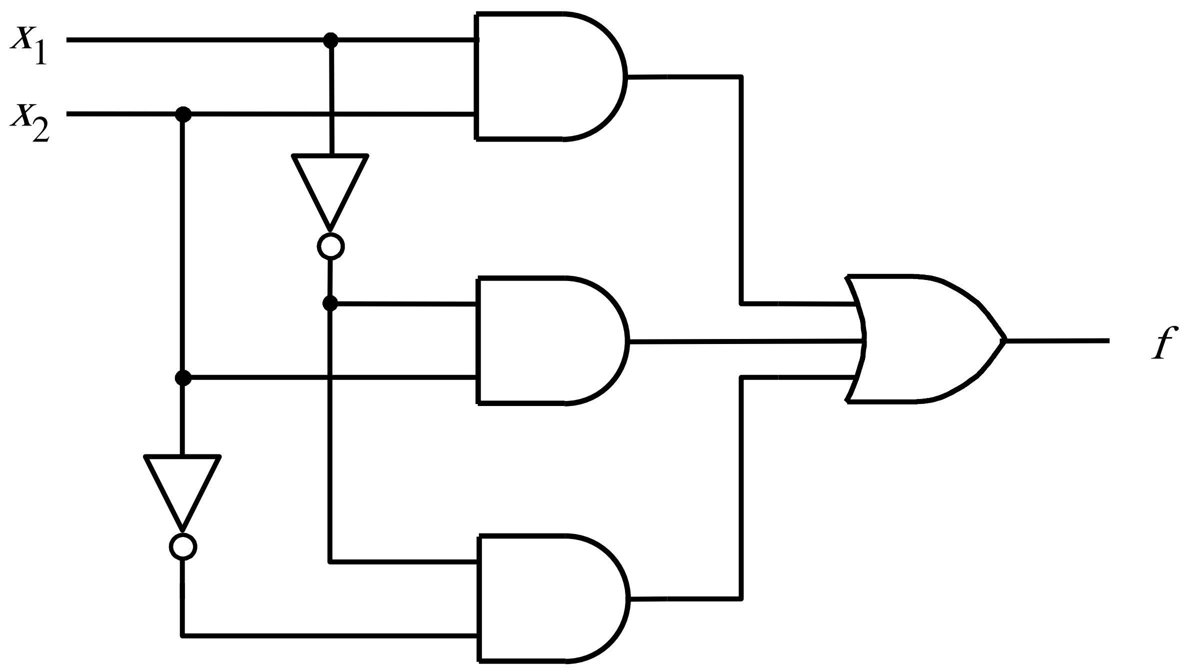
Minimal-cost realization

$$f(x_1, x_2) = x_2 + \overline{x}_1$$

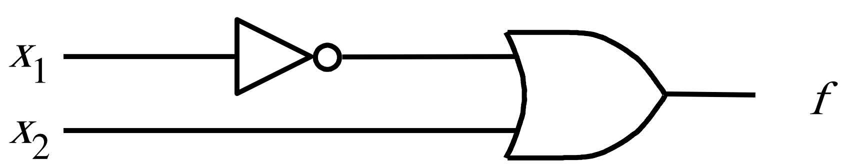


[Figure 2.20b from the textbook]

Two implementations for the same function



(a) Canonical sum-of-products



(b) Minimal-cost realization

[Figure 2.20 from the textbook]

Basis Functions
(for three variables)

Eight Basis Functions

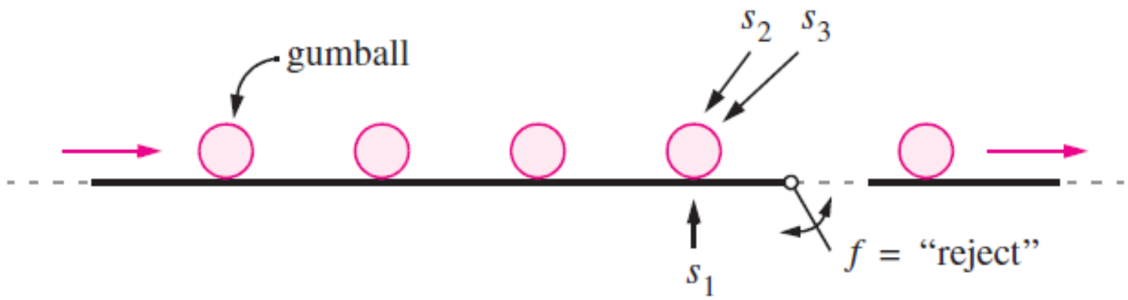
x	y	z	f_{000}	f_{001}	f_{010}	f_{011}	f_{100}	f_{101}	f_{110}	f_{111}
0	0	0	1	0	0	0	0	0	0	0
0	0	1	0	1	0	0	0	0	0	0
0	1	0	0	0	1	0	0	0	0	0
0	1	1	0	0	0	1	0	0	0	0
1	0	0	0	0	0	0	1	0	0	0
1	0	1	0	0	0	0	0	1	0	0
1	1	0	0	0	0	0	0	0	1	0
1	1	1	0	0	0	0	0	0	0	1

Eight Basis Functions

x	y	z	f_{000}	f_{001}	f_{010}	f_{011}	f_{100}	f_{101}	f_{110}	f_{111}
0	0	0	1	0	0	0	0	0	0	0
0	0	1	0	1	0	0	0	0	0	0
0	1	0	0	0	1	0	0	0	0	0
0	1	1	0	0	0	1	0	0	0	0
1	0	0	0	0	0	0	1	0	0	0
1	0	1	0	0	0	0	0	1	0	0
1	1	0	0	0	0	0	0	0	1	0
1	1	1	0	0	0	0	0	0	0	1

Function Synthesis Example
(with three variables)

Let's look at another problem



(a) Conveyor and sensors

s_1	s_2	s_3	f
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

(b) Truth table

[Figure 2.21 from the textbook]

Let's look at another problem

s_1	s_2	s_3	f
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

[Figure 2.21b from the textbook]

Let’s look at another problem

s_1	s_2	s_3	f
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

Let's look at another problem

s_1	s_2	s_3	f	
0	0	0	0	
0	0	1	1	$\bar{s}_1\bar{s}_2s_3$
0	1	0	0	
0	1	1	1	$\bar{s}_1s_2s_3$
1	0	0	0	
1	0	1	1	$s_1\bar{s}_2s_3$
1	1	0	1	$s_1s_2\bar{s}_3$
1	1	1	1	$s_1s_2s_3$

Let's look at another problem

s_1	s_2	s_3	f	
0	0	0	0	
0	0	1	1	$\overline{s}_1\overline{s}_2 s_3$
0	1	0	0	
0	1	1	1	$\overline{s}_1 s_2 s_3$
1	0	0	0	
1	0	1	1	$s_1\overline{s}_2 s_3$
1	1	0	1	$s_1 s_2\overline{s}_3$
1	1	1	1	$s_1 s_2 s_3$

Let’s look at another problem

s_1	s_2	s_3	f	
0	0	0	0	
0	0	1	1	$\bar{s}_1 \bar{s}_2 s_3$
0	1	0	0	
0	1	1	1	$\bar{s}_1 s_2 s_3$
1	0	0	0	
1	0	1	1	$s_1 \bar{s}_2 s_3$
1	1	0	1	$s_1 s_2 \bar{s}_3$
1	1	1	1	$s_1 s_2 s_3$

Let’s look at another problem

s_1	s_2	s_3	f	
0	0	0	0	
0	0	1	1	$\overline{s}_1 \overline{s}_2 s_3$
0	1	0	0	
0	1	1	1	$\overline{s}_1 s_2 s_3$
1	0	0	0	
1	0	1	1	$s_1 \overline{s}_2 s_3$
1	1	0	1	$s_1 s_2 \overline{s}_3$
1	1	1	1	$s_1 s_2 s_3$

Let's look at another problem

s_1	s_2	s_3	f	
0	0	0	0	
$\overline{0}$	$\overline{0}$	1	$\boxed{1}$	$\overline{s}_1 \overline{s}_2 s_3$
0	1	0	0	
$\overline{0}$	1	1	$\boxed{1}$	$\overline{s}_1 s_2 s_3$
1	0	0	0	
1	$\overline{0}$	1	$\boxed{1}$	$s_1 \overline{s}_2 s_3$
1	1	$\overline{0}$	$\boxed{1}$	$s_1 s_2 \overline{s}_3$
1	1	1	$\boxed{1}$	$s_1 s_2 s_3$

Let's look at another problem

s_1	s_2	s_3	f	
0	0	0	0	
0	0	1	1	$\bar{s}_1\bar{s}_2s_3$
0	1	0	0	
0	1	1	1	$\bar{s}_1s_2s_3$
1	0	0	0	
1	0	1	1	$s_1\bar{s}_2s_3$
1	1	0	1	$s_1s_2\bar{s}_3$
1	1	1	1	$s_1s_2s_3$

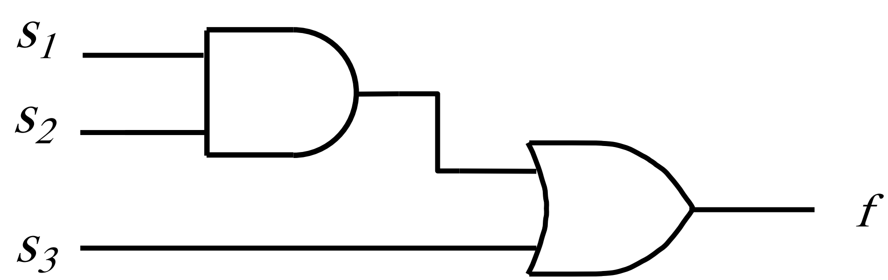
$$f = \bar{s}_1\bar{s}_2s_3 + \bar{s}_1s_2s_3 + s_1\bar{s}_2s_3 + s_1s_2\bar{s}_3 + s_1s_2s_3$$

Let's look at another problem (minimization)

$$
\begin{aligned}
f &= \bar{s}_1\bar{s}_2s_3 + \bar{s}_1s_2s_3 + s_1\bar{s}_2s_3 + s_1s_2s_3 + s_1s_2\bar{s}_3 + s_1s_2s_3 \\
&= \bar{s}_1s_3(\bar{s}_2 + s_2) + s_1s_3(\bar{s}_2 + s_2) + s_1s_2(\bar{s}_3 + s_3) \\
&= \bar{s}_1s_3 + s_1s_3 + s_1s_2 \\
&= s_3 + s_1s_2
\end{aligned}
$$

Let's look at another problem (minimization)

$$
\begin{aligned}
f &= \bar{s}_1\bar{s}_2s_3 + \bar{s}_1s_2s_3 + s_1\bar{s}_2s_3 + s_1s_2s_3 + s_1s_2\bar{s}_3 + s_1s_2s_3 \\
&= \bar{s}_1s_3(\bar{s}_2 + s_2) + s_1s_3(\bar{s}_2 + s_2) + s_1s_2(\bar{s}_3 + s_3) \\
&= \bar{s}_1s_3 + s_1s_3 + s_1s_2 \\
&= s_3 + s_1s_2
\end{aligned}
$$



Minterms and Maxterms

Row number	x_1	x_2	Minterm	Maxterm
0	0	0	$m_0 = \overline{x}_1\overline{x}_2$	$M_0 = x_1 + x_2$
1	0	1	$m_1 = \overline{x}_1 x_2$	$M_1 = x_1 + \overline{x}_2$
2	1	0	$m_2 = x_1\overline{x}_2$	$M_2 = \overline{x}_1 + x_2$
3	1	1	$m_3 = x_1 x_2$	$M_3 = \overline{x}_1 + \overline{x}_2$

Minterms and Maxterms

Row number	x_1	x_2	Minterm	Maxterm
0	0	0	$m_0 = \overline{x}_1\overline{x}_2$	$M_0 = x_1 + x_2$
1	0	1	$m_1 = \overline{x}_1 x_2$	$M_1 = x_1 + \overline{x}_2$
2	1	0	$m_2 = x_1\overline{x}_2$	$M_2 = \overline{x}_1 + x_2$
3	1	1	$m_3 = x_1 x_2$	$M_3 = \overline{x}_1 + \overline{x}_2$

Use these for
Sum-of-Products
Minimization
(1's of the function)

Use these for
Product-of-Sums
Minimization
(0's of the function)

Sum-of-Products Form

(uses the ones of the function)

Sum-of-Products Form
(for the AND logic function)

Row number	x_1	x_2	Minterm	$f(x_1, x_2)$
0	0	0	$m_0 = \overline{x}_1\overline{x}_2$	0
1	0	1	$m_1 = \overline{x}_1 x_2$	0
2	1	0	$m_2 = x_1\overline{x}_2$	0
3	1	1	$m_3 = x_1 x_2$	1

Sum-of-Products Form

(for the AND logic function)

Row number	x_1	x_2	Minterm	$f(x_1, x_2)$
0	0	0	$m_0 = \overline{x}_1\overline{x}_2$	0
1	0	1	$m_1 = \overline{x}_1 x_2$	0
2	1	0	$m_2 = x_1\overline{x}_2$	0
3	1	1	$m_3 = x_1 x_2$	1

Sum-of-Products Form

(for the AND logic function)

Row number	x_1	x_2	Minterm	$f(x_1, x_2)$
0	0	0	$m_0 = \overline{x}_1\overline{x}_2$	0
1	0	1	$m_1 = \overline{x}_1 x_2$	0
2	1	0	$m_2 = x_1\overline{x}_2$	0
3	1	1	$m_3 = x_1 x_2$	1

$$f(x_1, x_2) = m_3 = x_1\, x_2$$

(In this case there is just one product and there is no need for a sum)

Another Example

Sum-of-Products Form

Row number	x_1	x_2	Minterm	$f(x_1, x_2)$
0	0	0	$m_0 = \overline{x}_1\overline{x}_2$	1
1	0	1	$m_1 = \overline{x}_1 x_2$	1
2	1	0	$m_2 = x_1\overline{x}_2$	0
3	1	1	$m_3 = x_1 x_2$	1

Sum-of-Products Form

Row number	x_1	x_2	Minterm	$f(x_1, x_2)$
0	0	0	$m_0 = \overline{x}_1\overline{x}_2$	$\boxed{1}$
1	0	1	$m_1 = \overline{x}_1 x_2$	$\boxed{1}$
2	1	0	$m_2 = x_1\overline{x}_2$	0
3	1	1	$m_3 = x_1 x_2$	$\boxed{1}$

Sum-of-Products Form

Row number	x_1	x_2	Minterm	$f(x_1, x_2)$
0	0	0	$m_0 = \overline{x}_1\overline{x}_2$	1
1	0	1	$m_1 = \overline{x}_1 x_2$	1
2	1	0	$m_2 = x_1\overline{x}_2$	0
3	1	1	$m_3 = x_1 x_2$	1

$$
\begin{aligned}
f &= m_0 \cdot 1 + m_1 \cdot 1 + m_2 \cdot 0 + m_3 \cdot 1 \\
&= m_0 + m_1 + m_3 \\
&= \overline{x}_1\overline{x}_2 + \overline{x}_1 x_2 + x_1 x_2
\end{aligned}
$$

Product-of-Sums Form

(uses the zeros of the function)

Product-of-Sums Form
(for the OR logic function)

Row number	x_1	x_2	Maxterm	$f(x_1, x_2)$
0	0	0	$M_0 = x_1 + x_2$	0
1	0	1	$M_1 = x_1 + \overline{x}_2$	1
2	1	0	$M_2 = \overline{x}_1 + x_2$	1
3	1	1	$M_3 = \overline{x}_1 + \overline{x}_2$	1

Product-of-Sums Form
(for the OR logic function)

Row number	x_1	x_2	Maxterm	$f(x_1, x_2)$
0	0	0	$M_0 = x_1 + x_2$	$\boxed{0}$
1	0	1	$M_1 = x_1 + \overline{x}_2$	1
2	1	0	$M_2 = \overline{x}_1 + x_2$	1
3	1	1	$M_3 = \overline{x}_1 + \overline{x}_2$	1

Product-of-Sums Form
(for the OR logic function)

Row number	x_1	x_2	Maxterm	$f(x_1, x_2)$
0	0	0	$M_0 = x_1 + x_2$	0
1	0	1	$M_1 = x_1 + \overline{x}_2$	1
2	1	0	$M_2 = \overline{x}_1 + x_2$	1
3	1	1	$M_3 = \overline{x}_1 + \overline{x}_2$	1

$$f(x_1, x_2) = M_0 = x_1 + x_2$$

(In this case there is just one sum and there is no need for a product)

Another Example

Product-of-Sums Form
(for this logic function)

Row number	x_1	x_2	Maxterm	$f(x_1, x_2)$
0	0	0	$M_0 = x_1 + x_2$	0
1	0	1	$M_1 = x_1 + \overline{x_2}$	1
2	1	0	$M_2 = \overline{x_1} + x_2$	0
3	1	1	$M_3 = \overline{x_1} + \overline{x_2}$	1

Product-of-Sums Form

(for this logic function)

Row number	x_1	x_2	Maxterm	$f(x_1, x_2)$
0	0	0	$M_0 = x_1 + x_2$	0
1	0	1	$M_1 = x_1 + \overline{x}_2$	1
2	1	0	$M_2 = \overline{x}_1 + x_2$	0
3	1	1	$M_3 = \overline{x}_1 + \overline{x}_2$	1

Product-of-Sums Form
(for this logic function)

Row number	x_1	x_2	Maxterm	$f(x_1, x_2)$
0	0	0	$M_0 = x_1 + x_2$	0
1	0	1	$M_1 = x_1 + \overline{x}_2$	1
2	1	0	$M_2 = \overline{x}_1 + x_2$	0
3	1	1	$M_3 = \overline{x}_1 + \overline{x}_2$	1

$$f(x_1, x_2) = M_0 \bullet M_2 = (x_1 + x_2) \bullet (\overline{x}_1 + x_2)$$

Yet Another Example

Product-of-Sums Form

Row number	x_1	x_2	Maxterm	$f(x_1, x_2)$
0	0	0	$M_0 = x_1 + x_2$	1
1	0	1	$M_1 = x_1 + \overline{x}_2$	1
2	1	0	$M_2 = \overline{x}_1 + x_2$	$\boxed{0}$
3	1	1	$M_3 = \overline{x}_1 + \overline{x}_2$	1

We need to minimize using the zeros of the function f.
But let's first minimize the inverse of f, i.e., $\overline{\text{f}}$.

Product-of-Sums Form

Row number	x_1	x_2	Maxterm	$f(x_1, x_2)$	$\overline{f}(x_1, x_2)$
0	0	0	$M_0 = x_1 + x_2$	1	0
1	0	1	$M_1 = x_1 + \overline{x}_2$	1	0
2	1	0	$M_2 = \overline{x}_1 + x_2$	0	1
3	1	1	$M_3 = \overline{x}_1 + \overline{x}_2$	1	0

Product-of-Sums Form

Row number	x_1	x_2	Maxterm	$f(x_1, x_2)$	$\overline{f}(x_1, x_2)$
0	0	0	$M_0 = x_1 + x_2$	1	0
1	0	1	$M_1 = x_1 + \overline{x}_2$	1	0
2	1	0	$M_2 = \overline{x}_1 + x_2$	0	1
3	1	1	$M_3 = \overline{x}_1 + \overline{x}_2$	1	0

Product-of-Sums Form

Row number	x_1	x_2	Maxterm	$f(x_1, x_2)$	$\overline{f}(x_1, x_2)$
0	0	0	$M_0 = x_1 + x_2$	1	0
1	0	1	$M_1 = x_1 + \overline{x}_2$	1	0
2	1	0	$M_2 = \overline{x}_1 + x_2$	0	$\boxed{1}$
3	1	1	$M_3 = \overline{x}_1 + \overline{x}_2$	1	0

$$\overline{f}(x_1, x_2) = m_2 \\ = x_1\overline{x}_2$$

Product-of-Sums Form

Row number	x_1	x_2	Maxterm	$f(x_1, x_2)$	$\overline{f}(x_1, x_2)$
0	0	0	$M_0 = x_1 + x_2$	1	0
1	0	1	$M_1 = x_1 + \overline{x}_2$	1	0
2	1	0	$M_2 = \overline{x}_1 + x_2$	0	1
3	1	1	$M_3 = \overline{x}_1 + \overline{x}_2$	1	0

$$\overline{\overline{f}} = f = \overline{x_1 \overline{x}_2} = \overline{x}_1 + x_2$$

$$\overline{f}(x_1, x_2) = m_2 = x_1 \overline{x}_2$$

Product-of-Sums Form

Row number	x_1	x_2	Maxterm	$f(x_1, x_2)$	$\overline{f}(x_1, x_2)$
0	0	0	$M_0 = x_1 + x_2$	1	0
1	0	1	$M_1 = x_1 + \overline{x}_2$	1	0
2	1	0	$M_2 = \overline{x}_1 + x_2$	0	1
3	1	1	$M_3 = \overline{x}_1 + \overline{x}_2$	1	0

$$\overline{\overline{f}} = f = \overline{x_1\overline{x}_2} = \overline{x}_1 + x_2$$

$$\overline{f}(x_1, x_2) = m_2 = x_1\overline{x}_2$$

$$f = \overline{m}_2 = M_2$$

Examples with three-variable functions

Minterms and Maxterms (with three variables)

Row number	x_1	x_2	x_3	Minterm	Maxterm
0	0	0	0	$m_0 = \overline{x}_1\overline{x}_2\overline{x}_3$	$M_0 = x_1 + x_2 + x_3$
1	0	0	1	$m_1 = \overline{x}_1\overline{x}_2 x_3$	$M_1 = x_1 + x_2 + \overline{x}_3$
2	0	1	0	$m_2 = \overline{x}_1 x_2\overline{x}_3$	$M_2 = x_1 + \overline{x}_2 + x_3$
3	0	1	1	$m_3 = \overline{x}_1 x_2 x_3$	$M_3 = x_1 + \overline{x}_2 + \overline{x}_3$
4	1	0	0	$m_4 = x_1\overline{x}_2\overline{x}_3$	$M_4 = \overline{x}_1 + x_2 + x_3$
5	1	0	1	$m_5 = x_1\overline{x}_2 x_3$	$M_5 = \overline{x}_1 + x_2 + \overline{x}_3$
6	1	1	0	$m_6 = x_1 x_2\overline{x}_3$	$M_6 = \overline{x}_1 + \overline{x}_2 + x_3$
7	1	1	1	$m_7 = x_1 x_2 x_3$	$M_7 = \overline{x}_1 + \overline{x}_2 + \overline{x}_3$

[Figure 2.22 from the textbook]

A three-variable function

Row number	x_1	x_2	x_3	$f(x_1, x_2, x_3)$
0	0	0	0	0
1	0	0	1	1
2	0	1	0	0
3	0	1	1	0
4	1	0	0	1
5	1	0	1	1
6	1	1	0	1
7	1	1	1	0

[Figure 2.23 from the textbook]

Sum-of-Products (SOP)

Row number	x_1	x_2	x_3	$f(x_1, x_2, x_3)$
0	0	0	0	0
1	0	0	1	1
2	0	1	0	0
3	0	1	1	0
4	1	0	0	1
5	1	0	1	1
6	1	1	0	1
7	1	1	1	0

Sum-of-Products (SOP)

Row number	x_1	x_2	x_3	$f(x_1, x_2, x_3)$
0	0	0	0	0
1	0	0	1	1
2	0	1	0	0
3	0	1	1	0
4	1	0	0	1
5	1	0	1	1
6	1	1	0	1
7	1	1	1	0

$$f(x_1, x_2, x_3) = \bar{x}_1\bar{x}_2x_3 + x_1\bar{x}_2\bar{x}_3 + x_1\bar{x}_2x_3 + x_1x_2\bar{x}_3$$

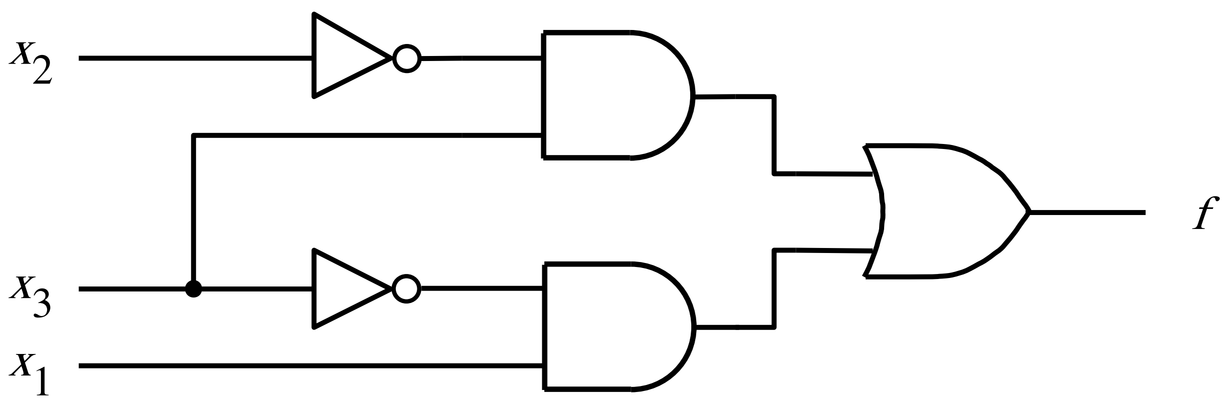
Sum-of-Products (SOP)

Row number	x_1	x_2	x_3	$f(x_1, x_2, x_3)$
0	0	0	0	0
1	0	0	1	1
2	0	1	0	0
3	0	1	1	0
4	1	0	0	1
5	1	0	1	1
6	1	1	0	1
7	1	1	1	0

$$f(x_1, x_2, x_3) = \bar{x}_1\bar{x}_2x_3 + x_1\bar{x}_2\bar{x}_3 + x_1\bar{x}_2x_3 + x_1x_2\bar{x}_3$$

$$\begin{aligned} f &= (\bar{x}_1 + x_1)\bar{x}_2x_3 + x_1(\bar{x}_2 + x_2)\bar{x}_3 \\ &= 1 \cdot \bar{x}_2x_3 + x_1 \cdot 1 \cdot \bar{x}_3 \\ &= \bar{x}_2x_3 + x_1\bar{x}_3 \end{aligned}$$

Sum-of-products realization of this function



$$f = \overline{x_2}\, x_3 \; + \; x_1 \overline{x_3}$$

[Figure 2.24a from the textbook]

A three-variable function

Row number	x_1	x_2	x_3	$f(x_1, x_2, x_3)$
0	0	0	0	0
1	0	0	1	1
2	0	1	0	0
3	0	1	1	0
4	1	0	0	1
5	1	0	1	1
6	1	1	0	1
7	1	1	1	0

[Figure 2.23 from the textbook]

Product-of-Sums (POS)

Row number	x_1	x_2	x_3	$f(x_1, x_2, x_3)$
0	0	0	0	$\boxed{0}$
1	0	0	1	1
2	0	1	0	$\boxed{0}$
3	0	1	1	$\boxed{0}$
4	1	0	0	1
5	1	0	1	1
6	1	1	0	1
7	1	1	1	$\boxed{0}$

Product-of-Sums (POS)

Row number	x_1	x_2	x_3	$f(x_1, x_2, x_3)$
0	0	0	0	$\boxed{0}$
1	0	0	1	1
2	0	1	0	$\boxed{0}$
3	0	1	1	$\boxed{0}$
4	1	0	0	1
5	1	0	1	1
6	1	1	0	1
7	1	1	1	$\boxed{0}$

$$
\begin{aligned}
f &= \overline{m_0 + m_2 + m_3 + m_7} \\
&= \overline{m}_0 \cdot \overline{m}_2 \cdot \overline{m}_3 \cdot \overline{m}_7 \\
&= M_0 \cdot M_2 \cdot M_3 \cdot M_7 \\
&= (x_1 + x_2 + x_3)(x_1 + \overline{x}_2 + x_3)(x_1 + \overline{x}_2 + \overline{x}_3)(\overline{x}_1 + \overline{x}_2 + \overline{x}_3)
\end{aligned}
$$

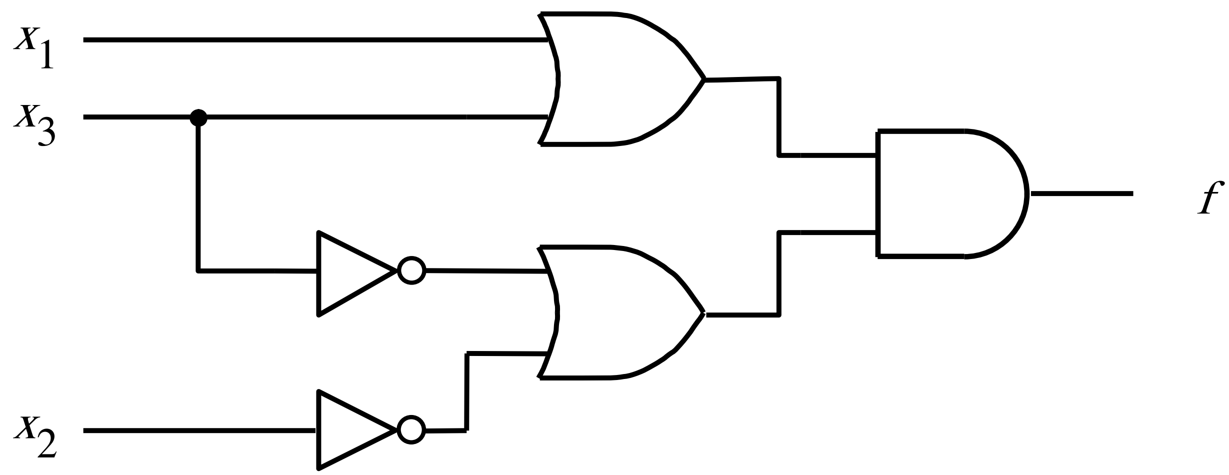
Product-of-Sums (POS)

Row number	x_1	x_2	x_3	$f(x_1, x_2, x_3)$
0	0	0	0	0
1	0	0	1	1
2	0	1	0	0
3	0	1	1	0
4	1	0	0	1
5	1	0	1	1
6	1	1	0	1
7	1	1	1	0

$$f = ((x_1 + x_3) + x_2)((x_1 + x_3) + \overline{x}_2)(x_1 + (\overline{x}_2 + \overline{x}_3))(\overline{x}_1 + (\overline{x}_2 + \overline{x}_3))$$

$$f = (x_1 + x_3)(\overline{x}_2 + \overline{x}_3)$$

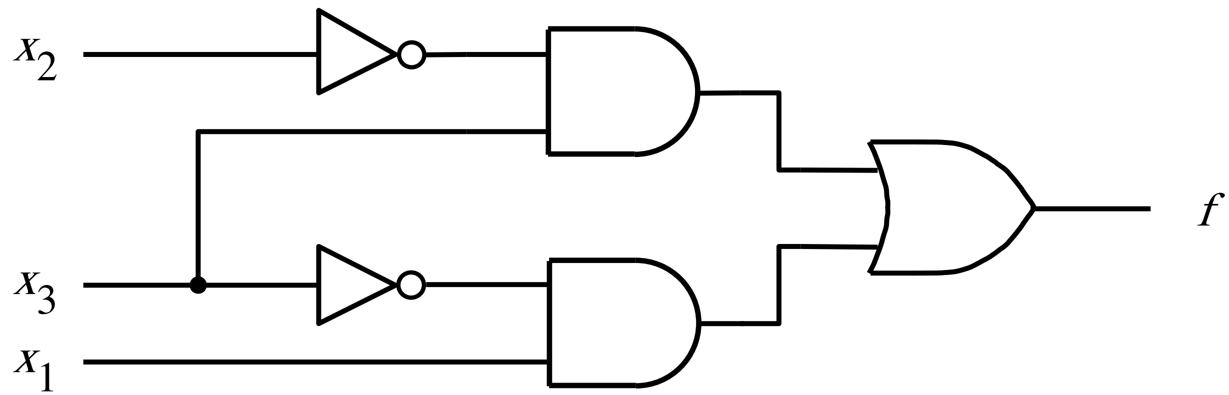
Product-of-sums realization of this function



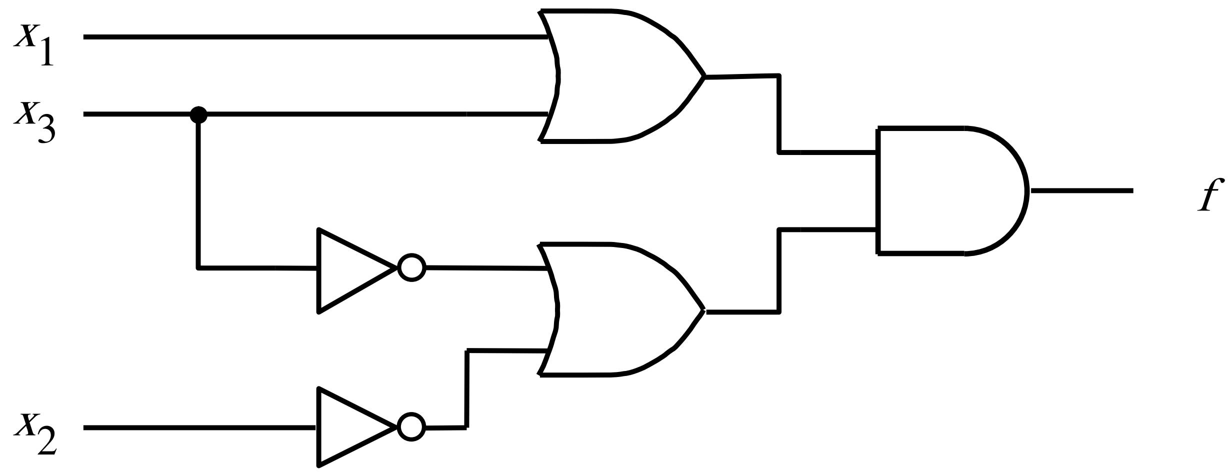
$$f = (x_1 + x_3) \bullet (\overline{x_2} + \overline{x_3})$$

[Figure 2.24b from the textbook]

Two realizations of this function



(a) A minimal sum-of-products realization



(b) A minimal product-of-sums realization

[Figure 2.24 from the textbook]

Shorthand Notation for SOP

Row number	x_1	x_2	x_3	$f(x_1, x_2, x_3)$
0	0	0	0	0
1	0	0	1	1
2	0	1	0	0
3	0	1	1	0
4	1	0	0	1
5	1	0	1	1
6	1	1	0	1
7	1	1	1	0

$$f(x_1, x_2, x_3) = \sum (m_1, m_4, m_5, m_6)$$

or

$$f(x_1, x_2, x_3) = \sum m(1, 4, 5, 6)$$

Shorthand Notation for POS

Row number	x_1	x_2	x_3	$f(x_1, x_2, x_3)$
0	0	0	0	$\boxed{0}$
1	0	0	1	1
2	0	1	0	$\boxed{0}$
3	0	1	1	$\boxed{0}$
4	1	0	0	1
5	1	0	1	1
6	1	1	0	1
7	1	1	1	$\boxed{0}$

$$f(x_1, x_2, x_3) = \Pi(M_0, M_2, M_3, M_7)$$

or

$$f(x_1, x_2, x_3) = \Pi M(0, 2, 3, 7)$$

Shorthand Notation

- **Sum-of-Products (SOP)**

$$f(x_1, x_2, x_3) = \sum(m_1, m_4, m_5, m_6)$$

or

$$f(x_1, x_2, x_3) = \sum m(1, 4, 5, 6)$$

- **Product-of-Sums (POS)**

$$f(x_1, x_2, x_3) = \Pi(M_0, M_2, M_3, M_7)$$

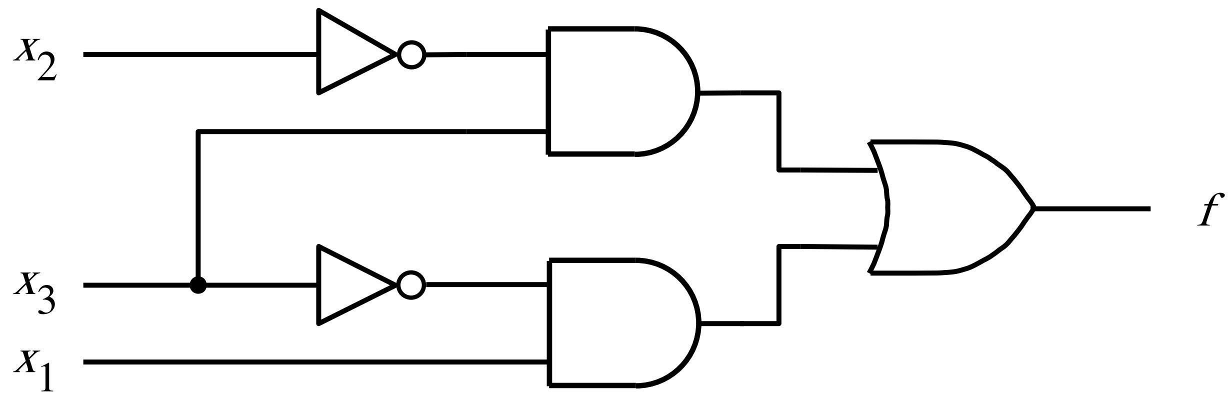
or

$$f(x_1, x_2, x_3) = \Pi M(0, 2, 3, 7)$$

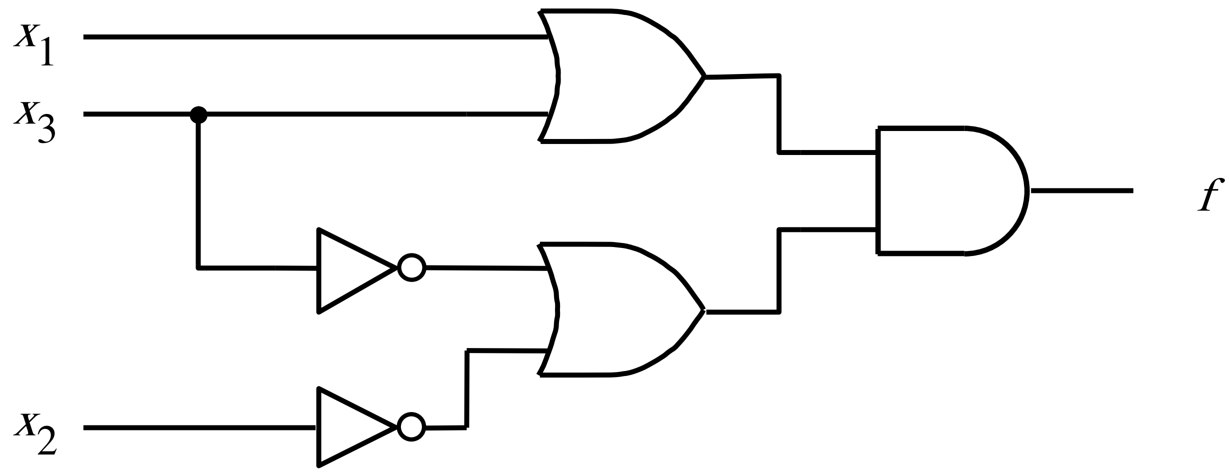
The Cost of a Circuit

- Count all gates
- Count all inputs/wires to the gates
- Add the two partial counts. That is the cost.

What is the cost of each circuit?

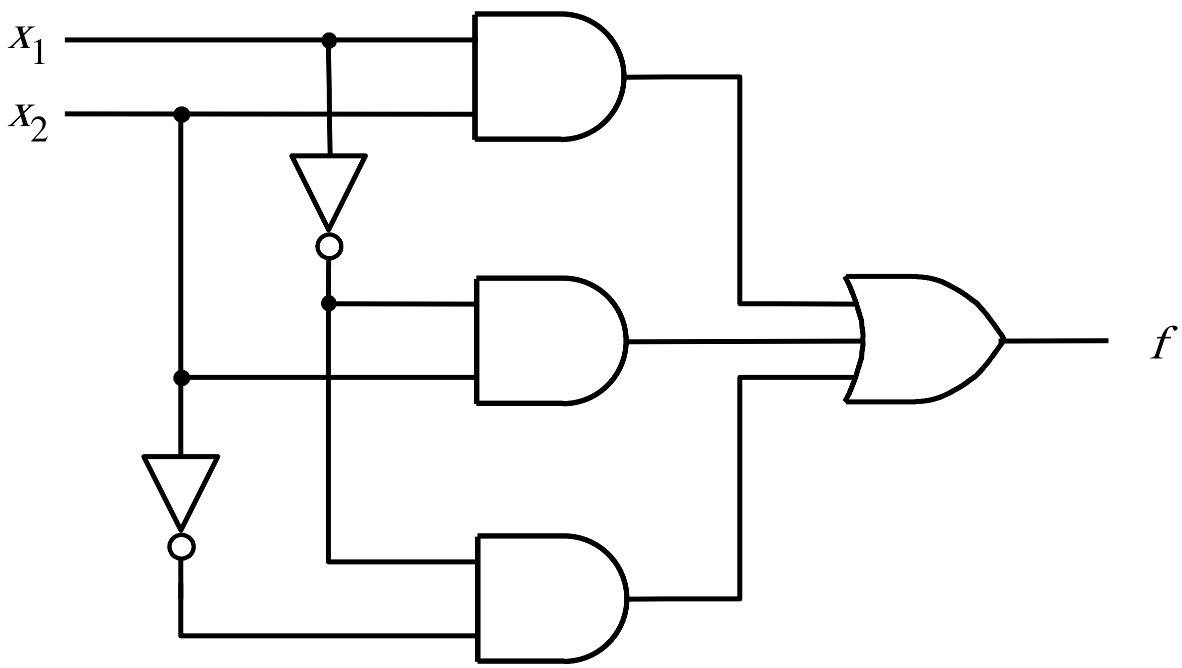


(a) A minimal sum-of-products realization



(b) A minimal product-of-sums realization

What is the cost of this circuit?



Questions?

THE END