

# **CprE 281: Digital Logic**

**Instructor: Alexander Stoytchev**

**<http://www.ece.iastate.edu/~alexs/classes/>**

# Signed Numbers

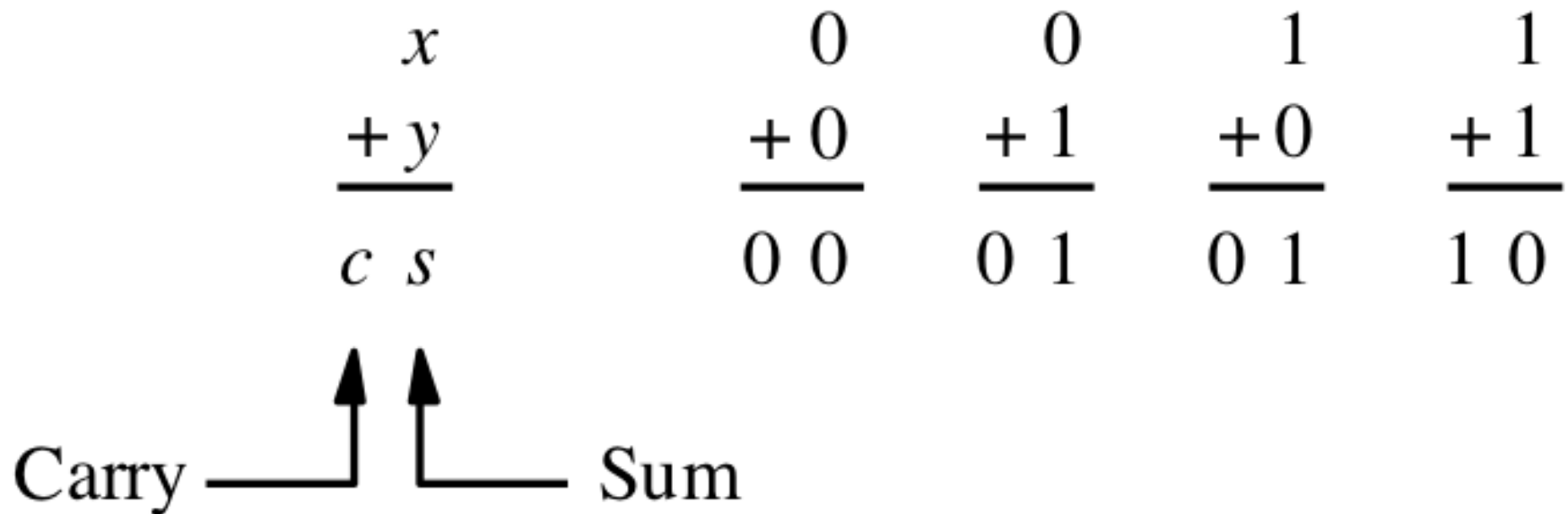
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# Signed **Integer** Numbers

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# Quick Review

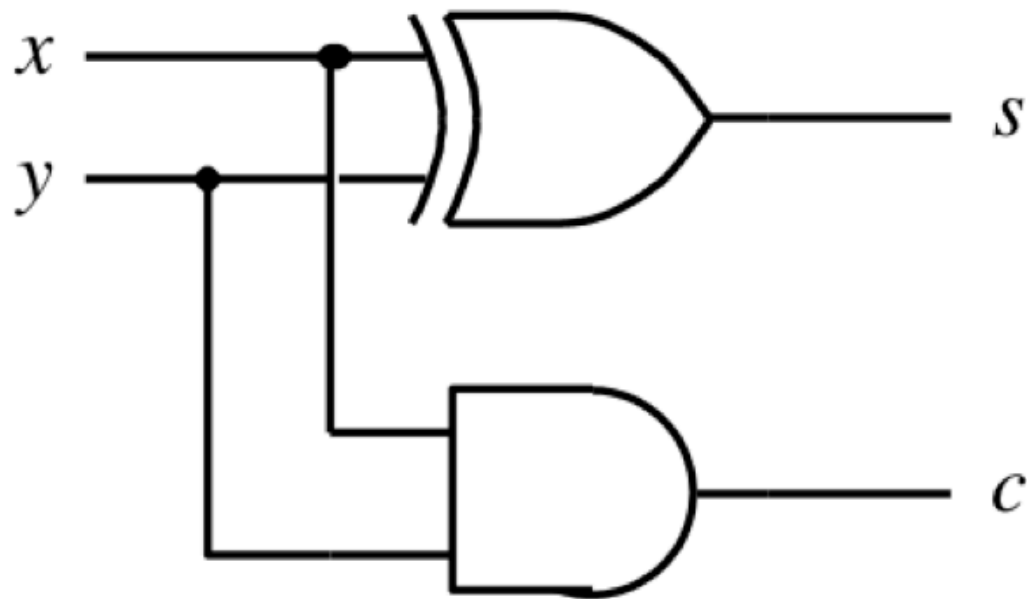
# Adding two bits (there are four possible cases)



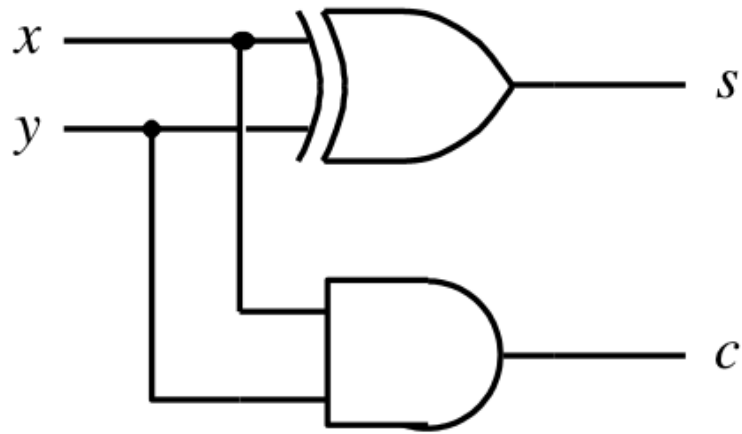
# Adding two bits (the truth table)

$x$	$y$	Carry $c$	Sum $s$
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

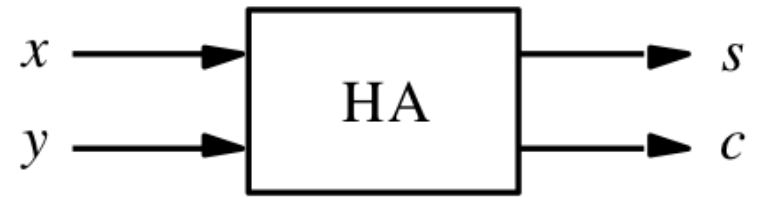
# Adding two bits (the logic circuit)



# The Half-Adder



(c) Circuit



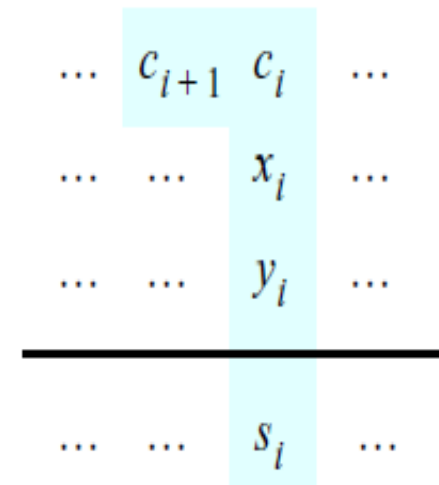
(d) Graphical symbol



# Addition of multibit numbers

Generated carries  $\longrightarrow$  1 1 1 0

$$\begin{array}{r}
 X = x_4x_3x_2x_1x_0 \quad 01111 \quad (15)_{10} \\
 + Y = y_4y_3y_2y_1y_0 \quad +01010 \quad + (10)_{10} \\
 \hline
 S = s_4s_3s_2s_1s_0 \quad 11001 \quad (25)_{10}
 \end{array}$$



Bit position  $i$

# Analogy with addition in base 10

$$\begin{array}{r} + \quad \quad \quad X_2 \quad X_1 \quad X_0 \\ \quad \quad \quad Y_2 \quad Y_1 \quad Y_0 \\ \hline \quad \quad \quad S_2 \quad S_1 \quad S_0 \end{array}$$

# Analogy with addition in base 10

$$\begin{array}{r} + \quad 3 \quad 8 \quad 9 \\ \quad 1 \quad 5 \quad 7 \\ \hline \quad 5 \quad 4 \quad 6 \end{array}$$

# Analogy with addition in base 10

carry	0	1	1	0
		3	8	9
+		1	5	7
		<hr/>		
		5	4	6

# Analogy with addition in base 10

$$\begin{array}{rcccc} & C_3 & C_2 & C_1 & C_0 \\ + & & X_2 & X_1 & X_0 \\ & & Y_2 & Y_1 & Y_0 \\ \hline & & S_2 & S_1 & S_0 \end{array}$$

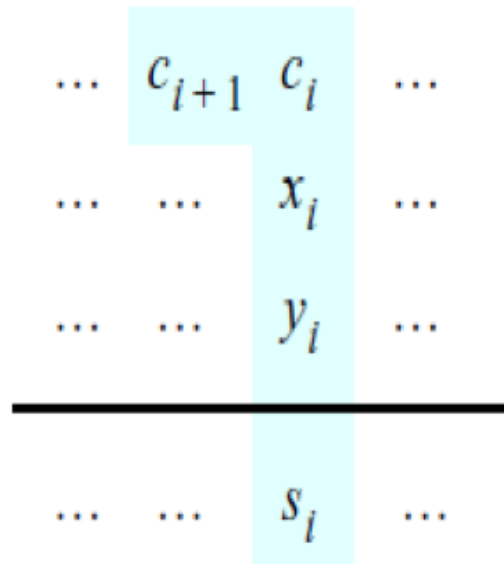
# Another example in base 10

$$\begin{array}{r} + \quad \quad \quad 9 \quad 3 \quad 8 \\ \quad \quad \quad 2 \quad 1 \quad 4 \\ \hline \quad \quad \quad 1 \quad 1 \quad 5 \quad 2 \end{array}$$

# Another example in base 10

carry	1	0	1	0
		9	3	8
+		2	1	4
		<hr/>		
		1	5	2

# Problem Statement and Truth Table



$c_i$	$x_i$	$y_i$	$c_{i+1}$	$s_i$
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

[ Figure 3.2b from the textbook ]

[ Figure 3.3a from the textbook ]



# Let's fill-in the two K-maps

$c_i$	$x_i$	$y_i$	$c_{i+1}$	$s_i$
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

	$x_i y_i$	00	01	11	10
$c_i$	0				
	1				

$s_i =$

	$x_i y_i$	00	01	11	10
$c_i$	0				
	1				

$c_{i+1} =$

# Let's fill-in the two K-maps

$c_i$	$x_i$	$y_i$	$c_{i+1}$	$s_i$
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

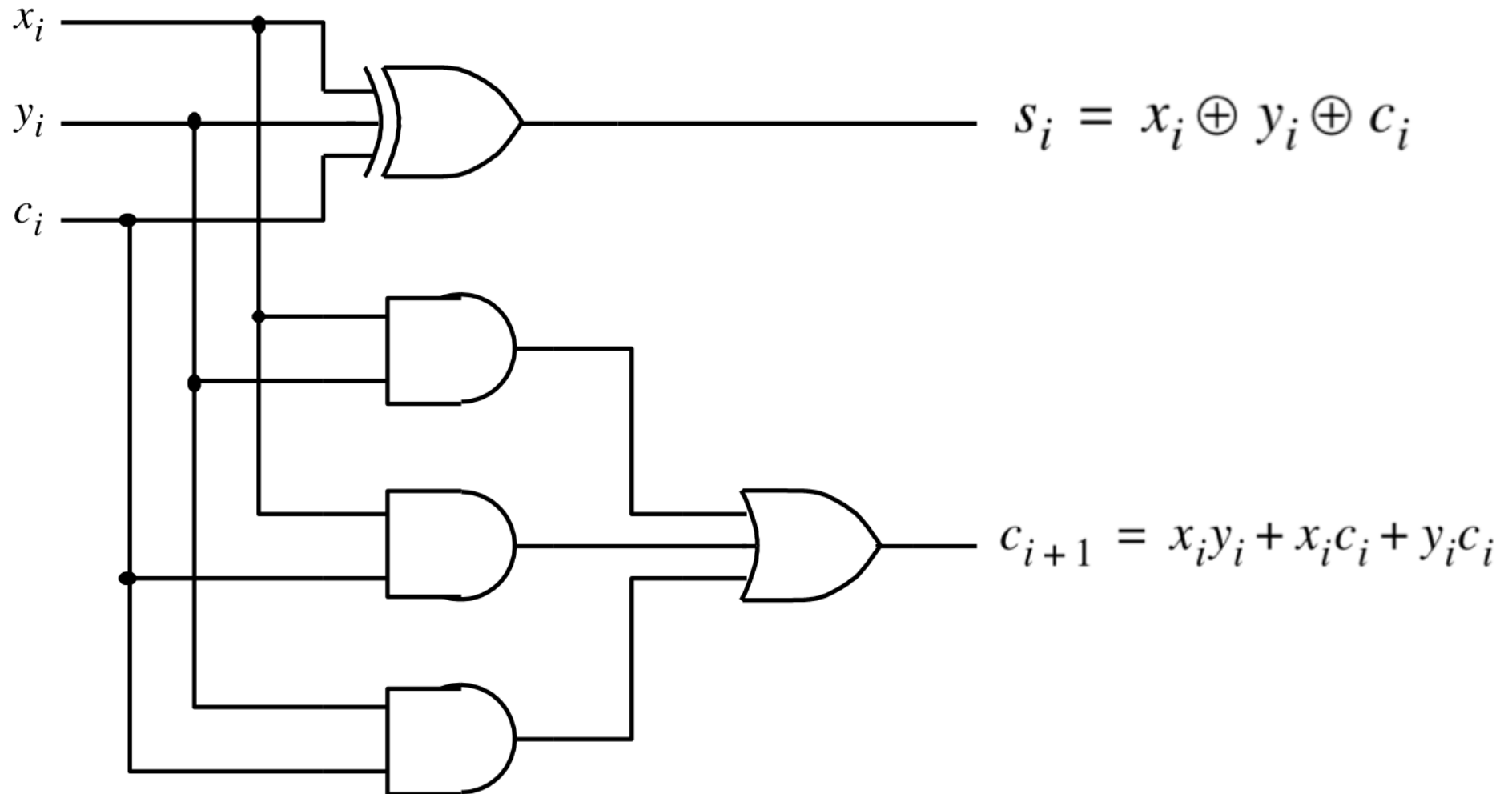
	$x_i y_i$				
$c_i$		00	01	11	10
0			1		1
1		1		1	

$$s_i = x_i \oplus y_i \oplus c_i$$

	$x_i y_i$				
$c_i$		00	01	11	10
0				1	
1			1	1	1

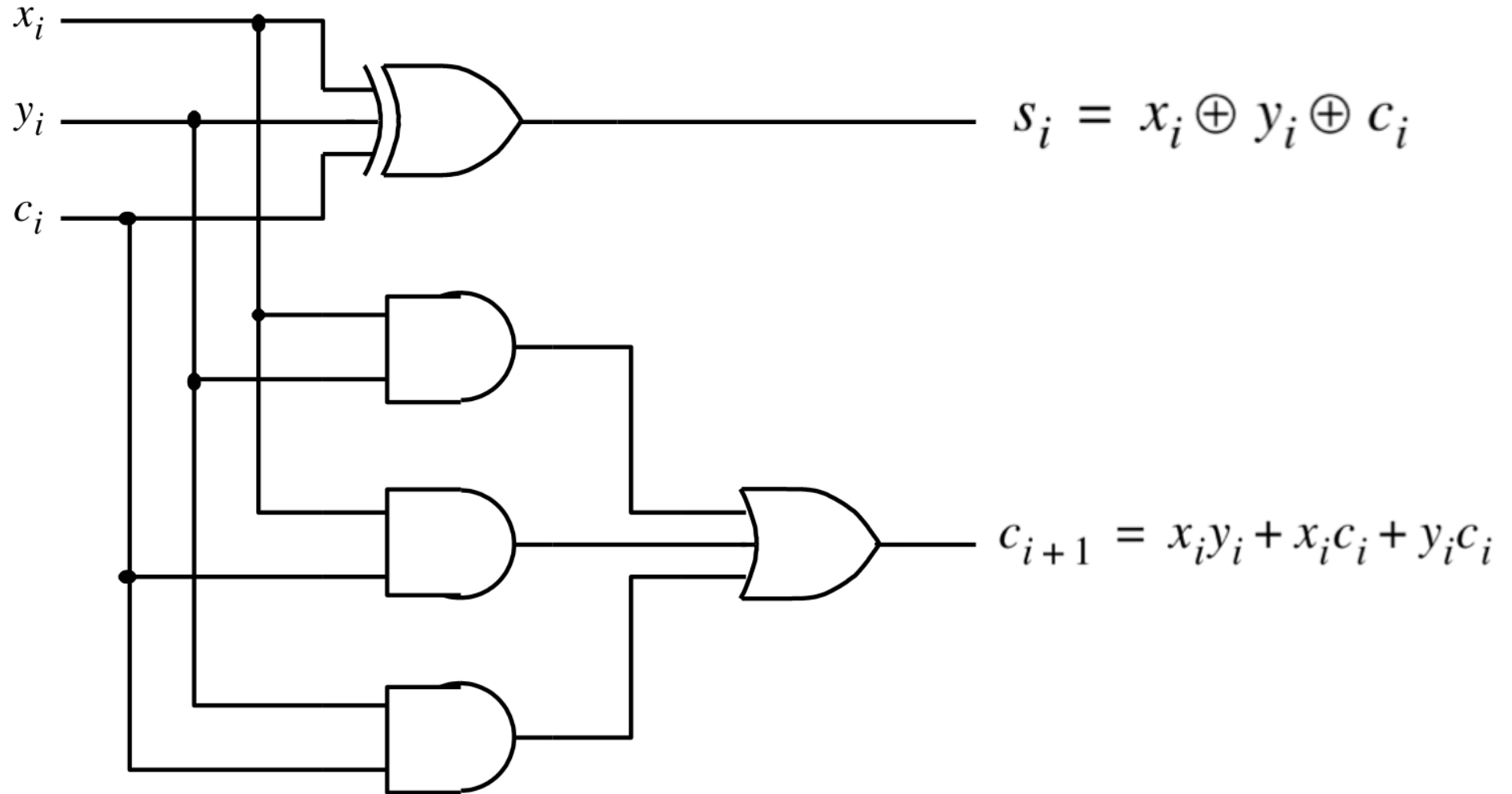
$$c_{i+1} = x_i y_i + x_i c_i + y_i c_i$$

# The circuit for the two expressions



[ Figure 3.3c from the textbook ]

# This is called the Full-Adder

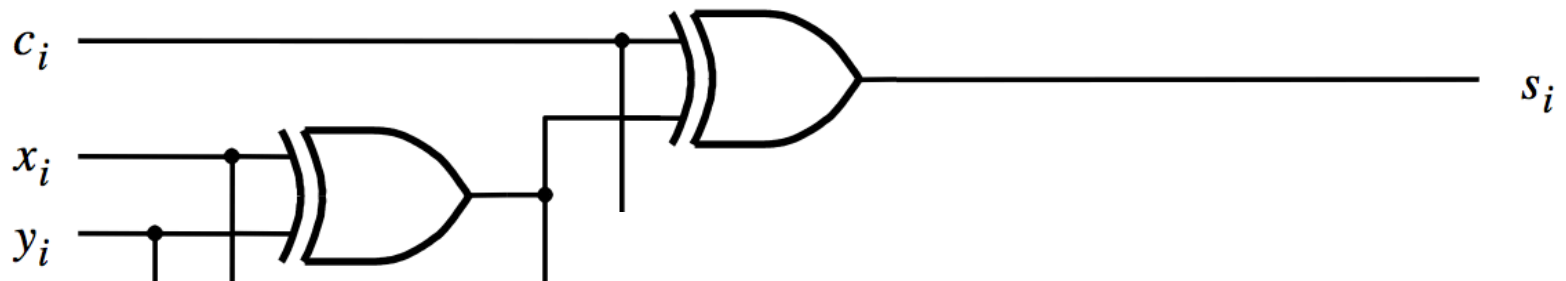
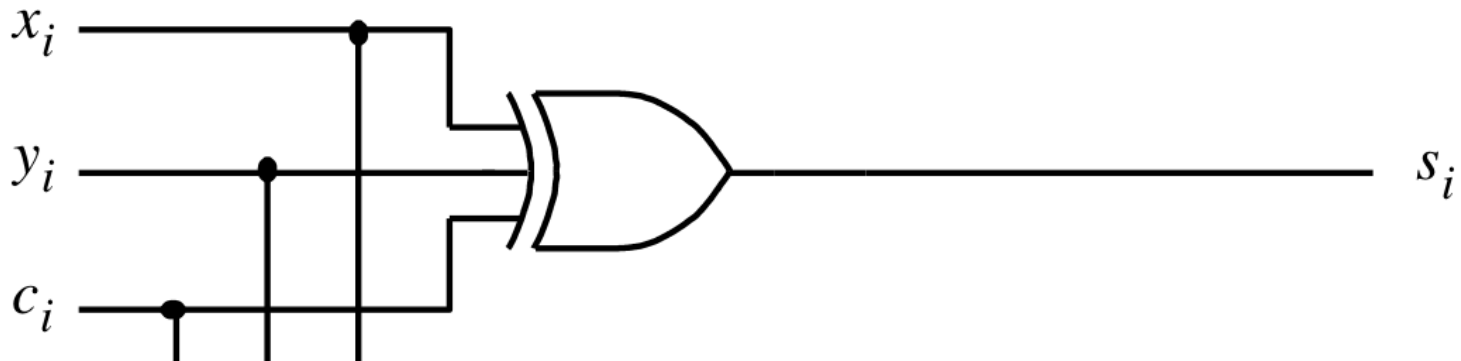


[ Figure 3.3c from the textbook ]

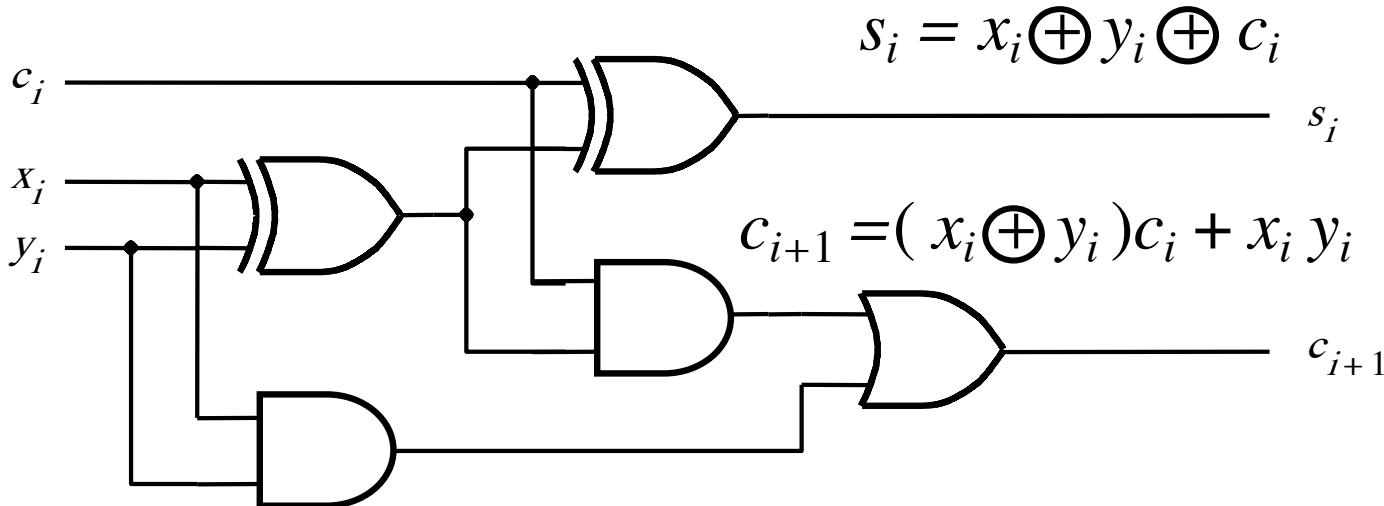
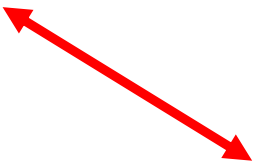
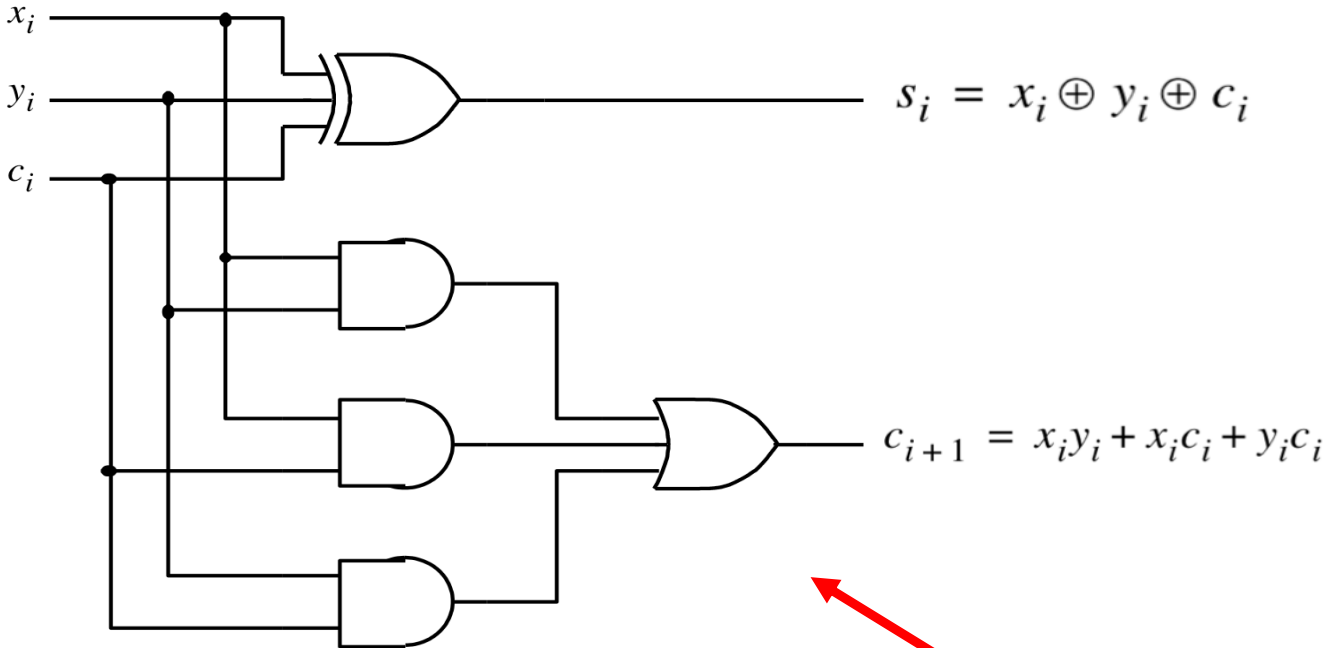
# XOR Magic

( $s_i$  can be implemented in two different ways)

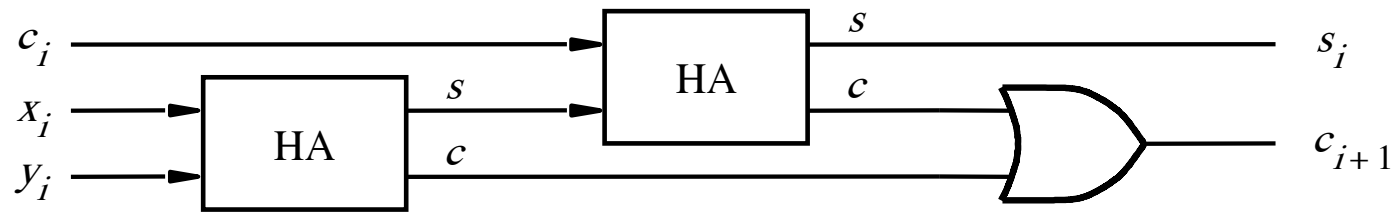
$$s_i = x_i \oplus y_i \oplus c_i$$



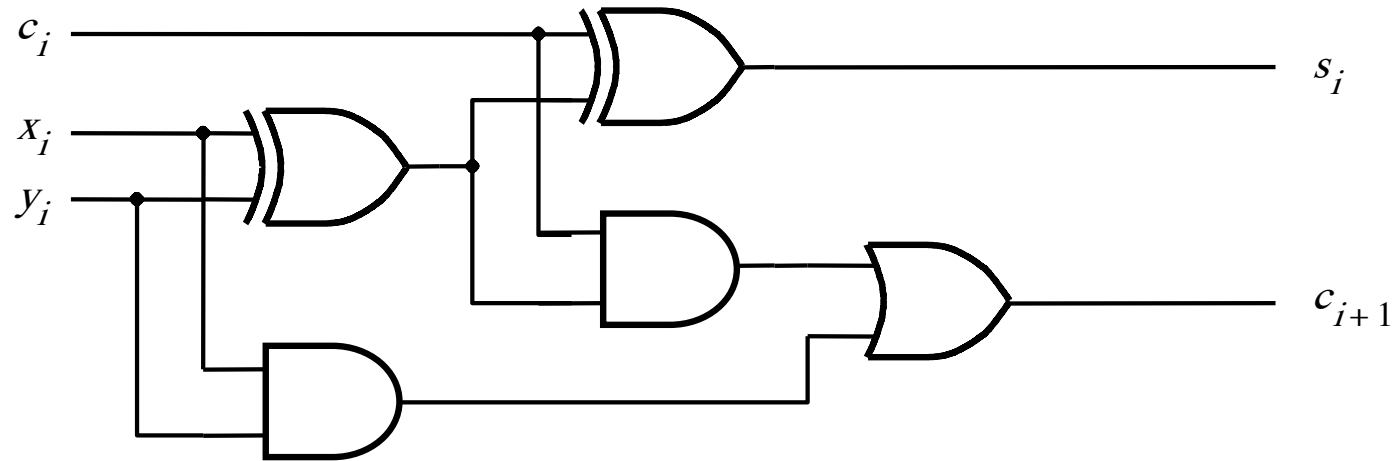
# These two circuits are equivalent



# A decomposed implementation of the full-adder circuit

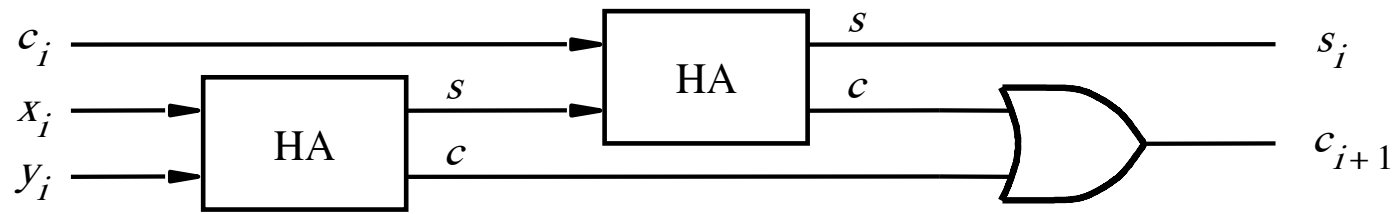


(a) Block diagram

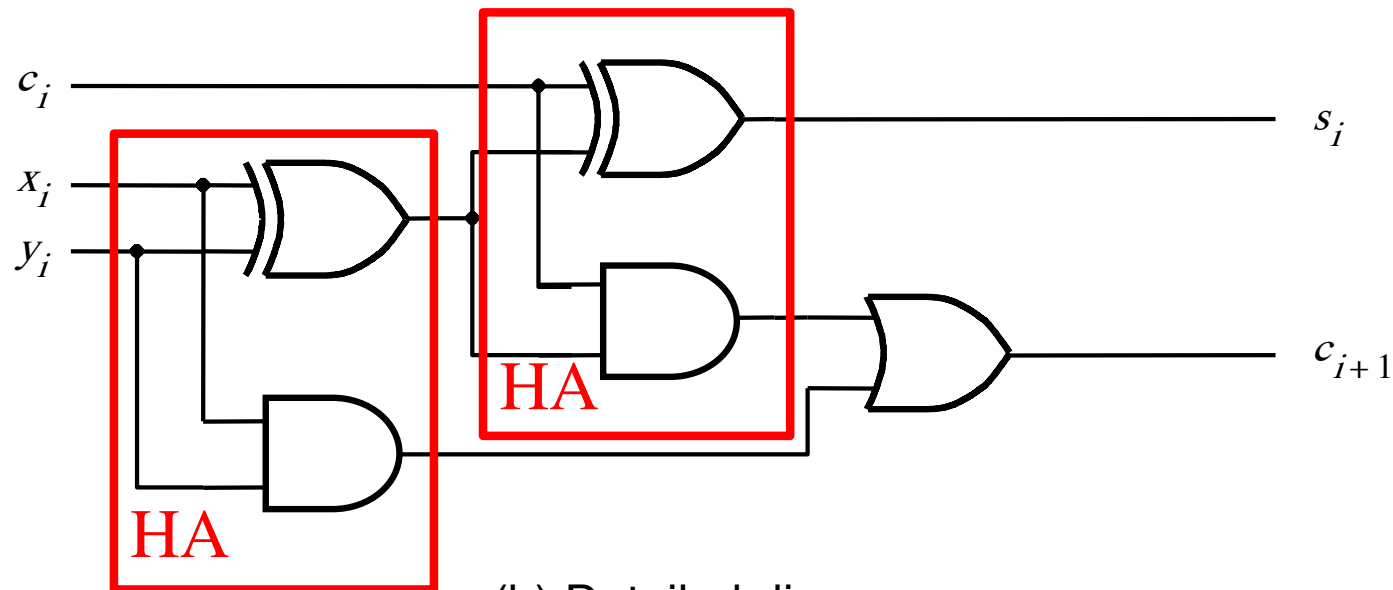


(b) Detailed diagram

# A decomposed implementation of the full-adder circuit



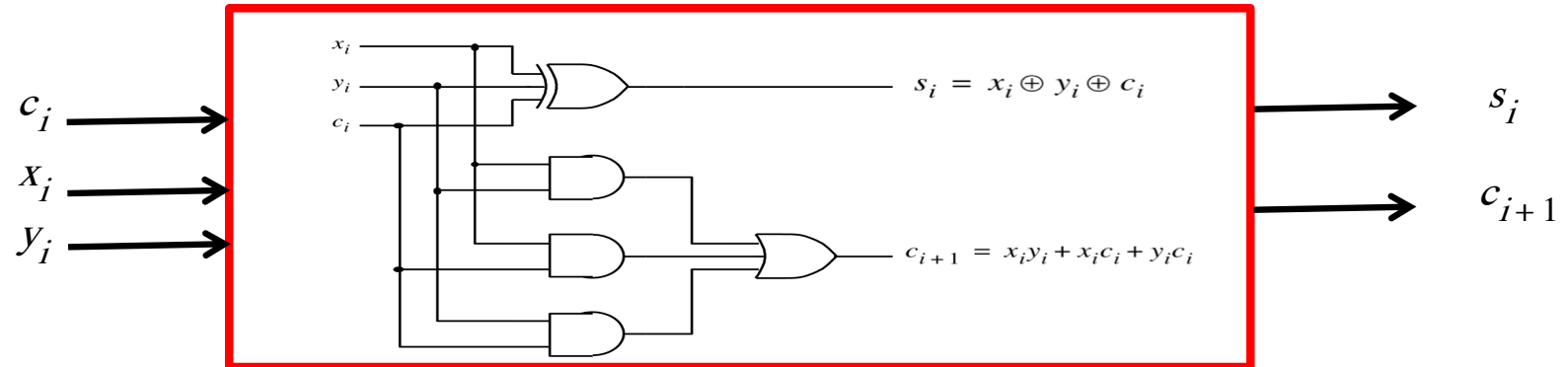
(a) Block diagram



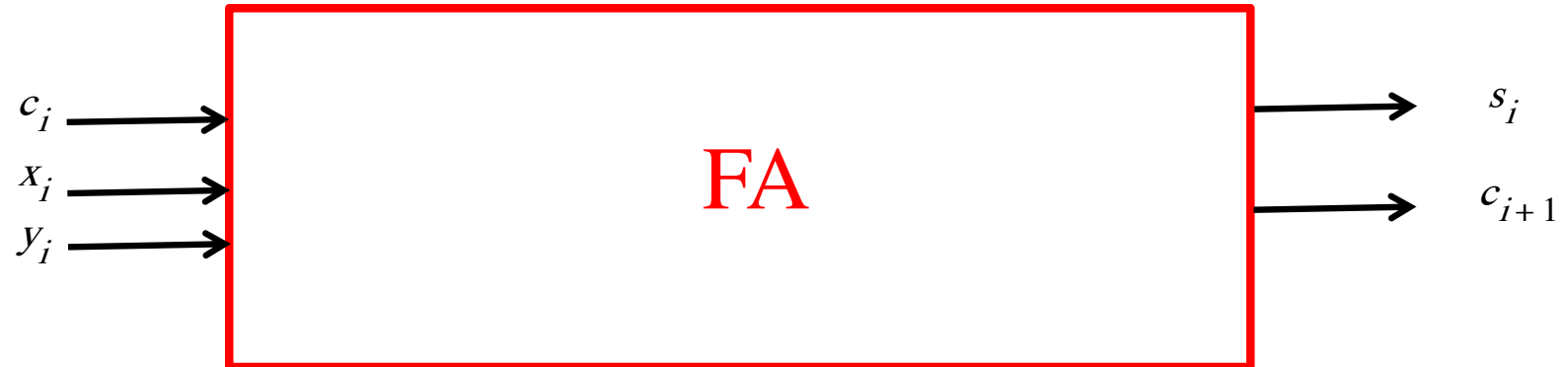
(b) Detailed diagram



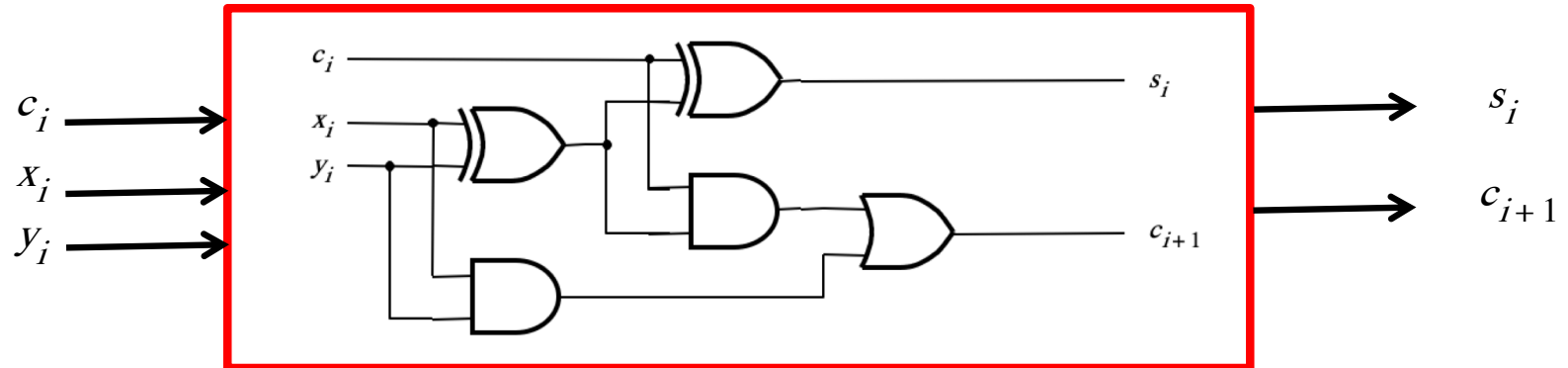
# The Full-Adder Abstraction



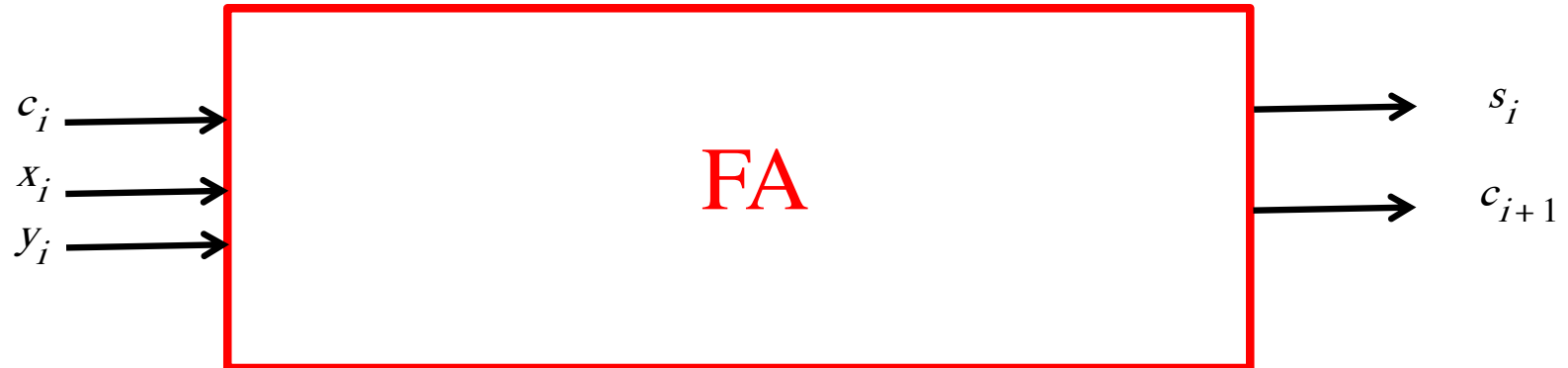
# The Full-Adder Abstraction



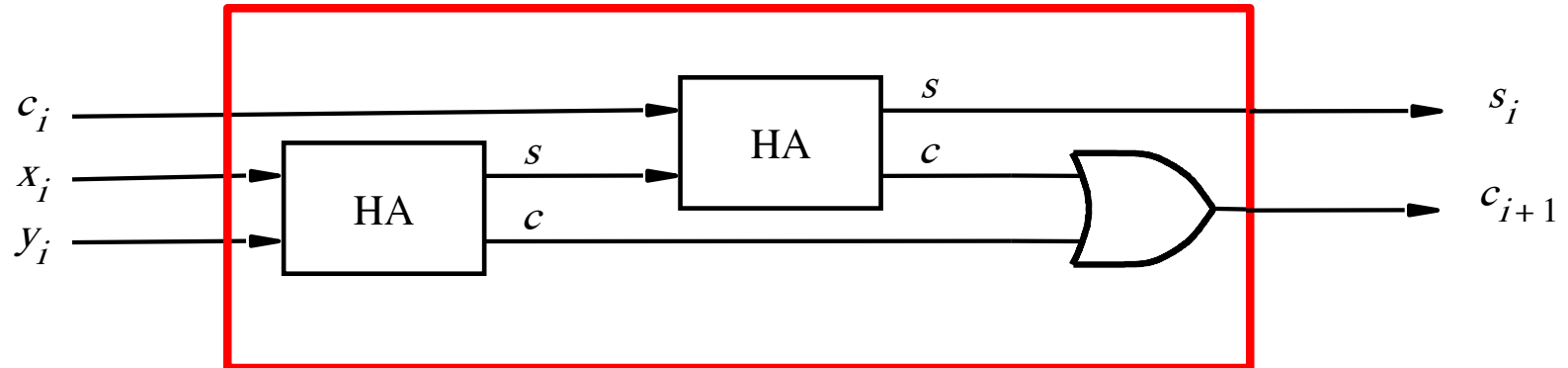
# The Full-Adder Abstraction



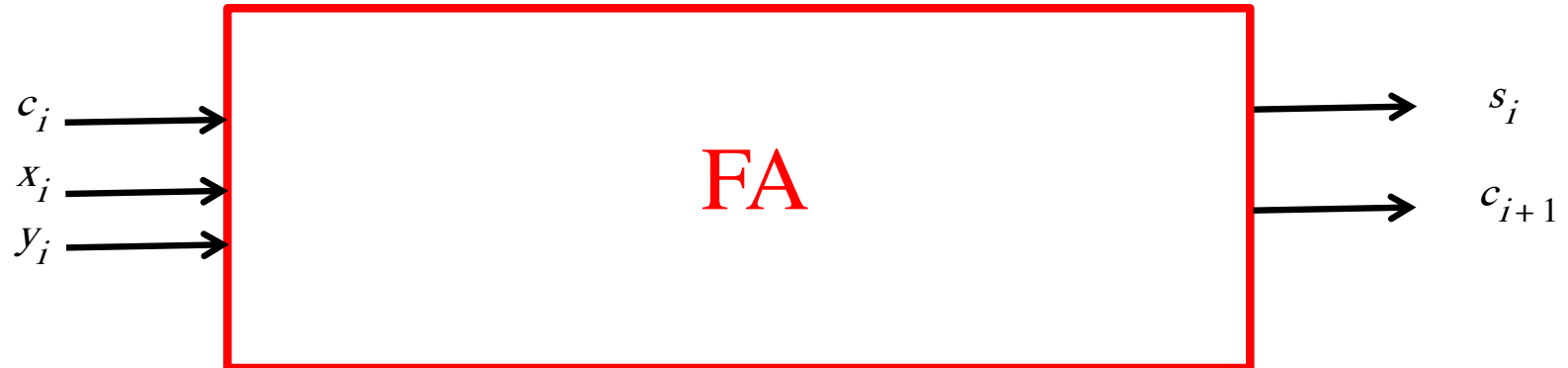
# The Full-Adder Abstraction



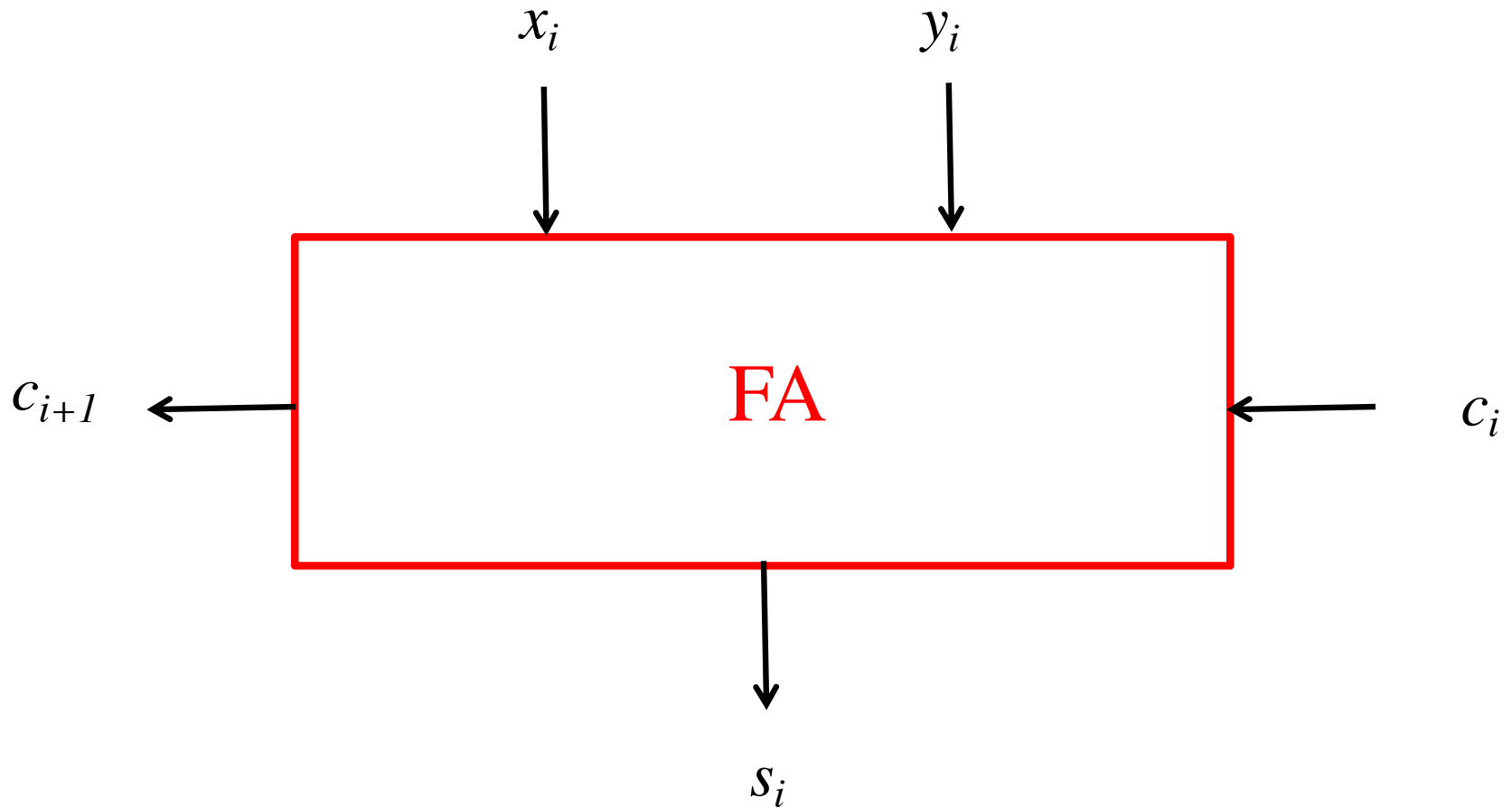
# The Full-Adder Abstraction



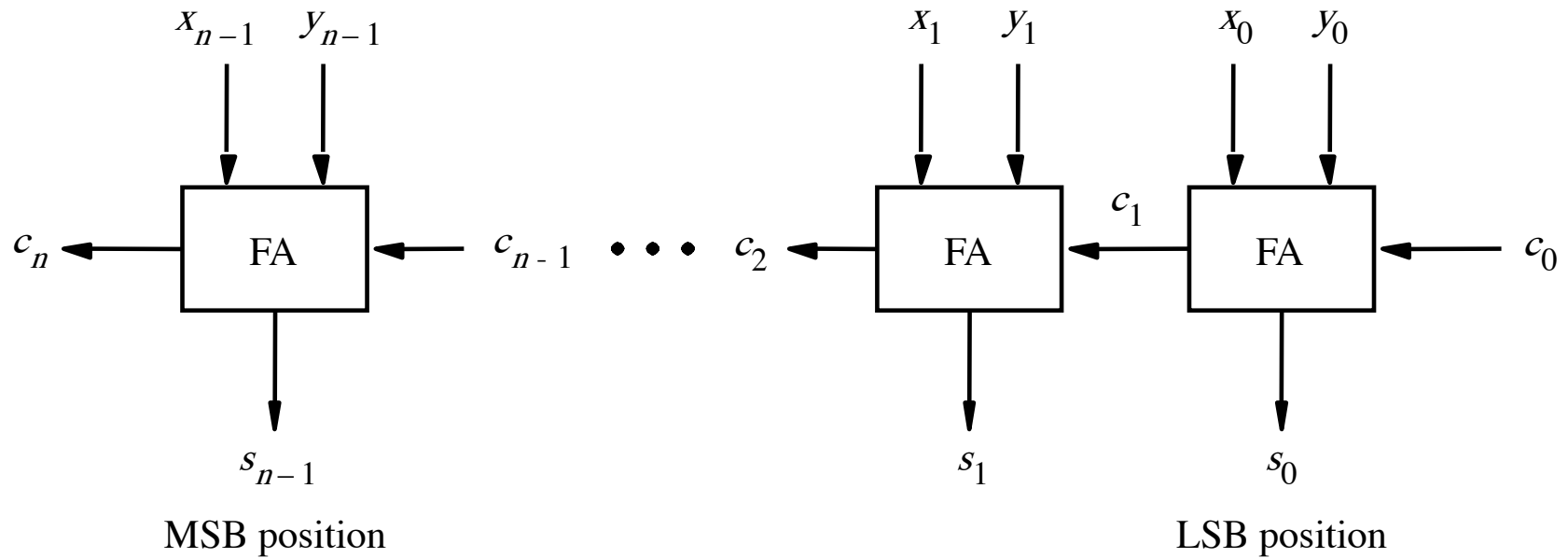
# The Full-Adder Abstraction



**We can place the arrows anywhere**



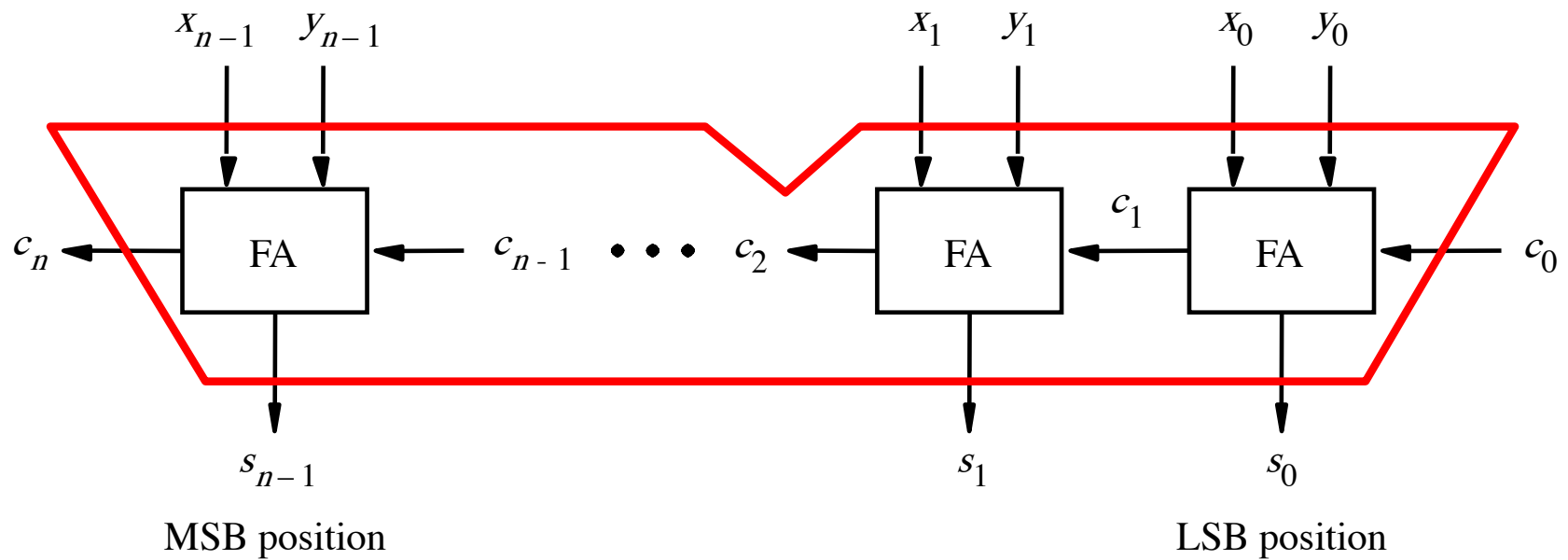
# *n*-bit ripple-carry adder



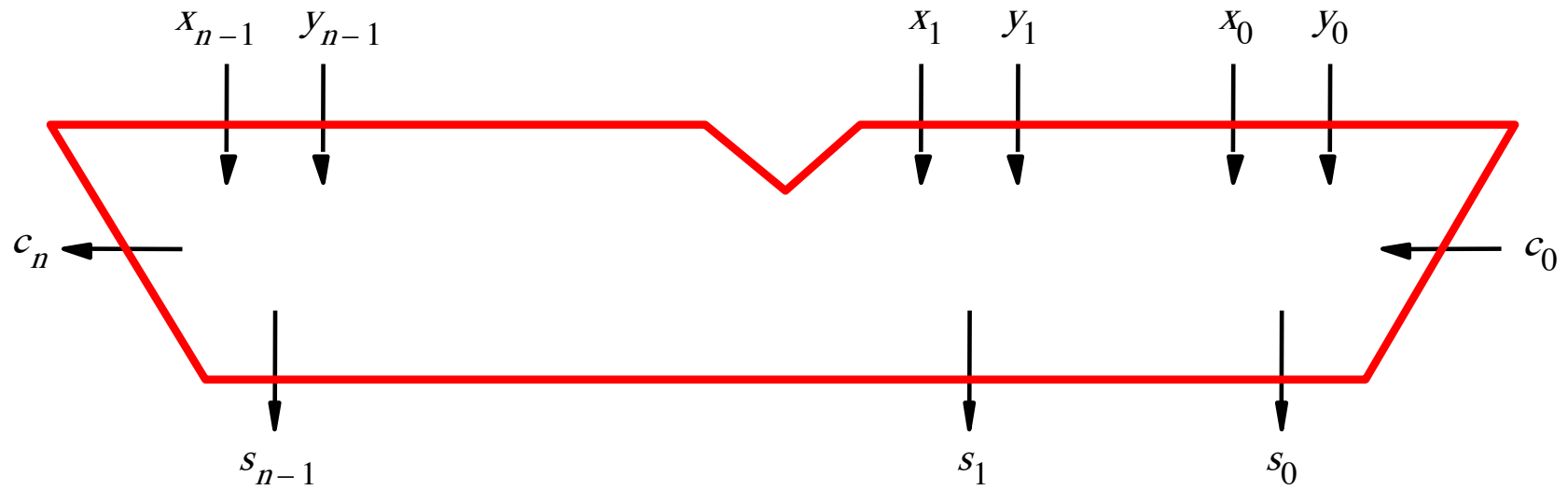
[ Figure 3.5 from the textbook ]



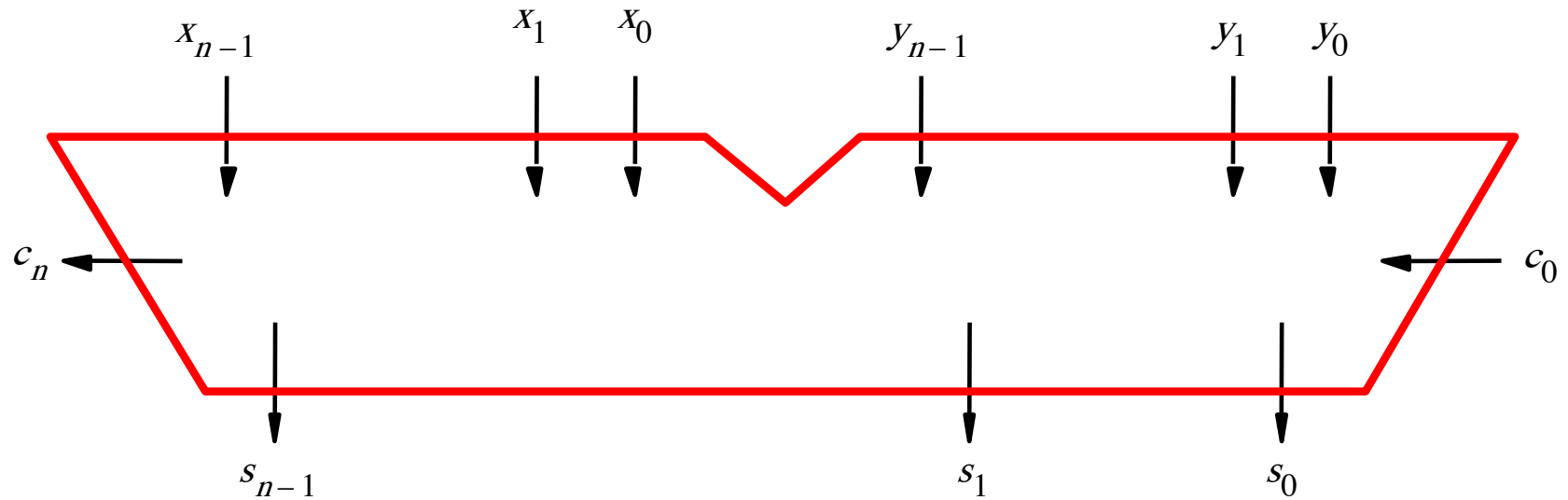
# *n*-bit ripple-carry adder abstraction



# *n*-bit ripple-carry adder abstraction

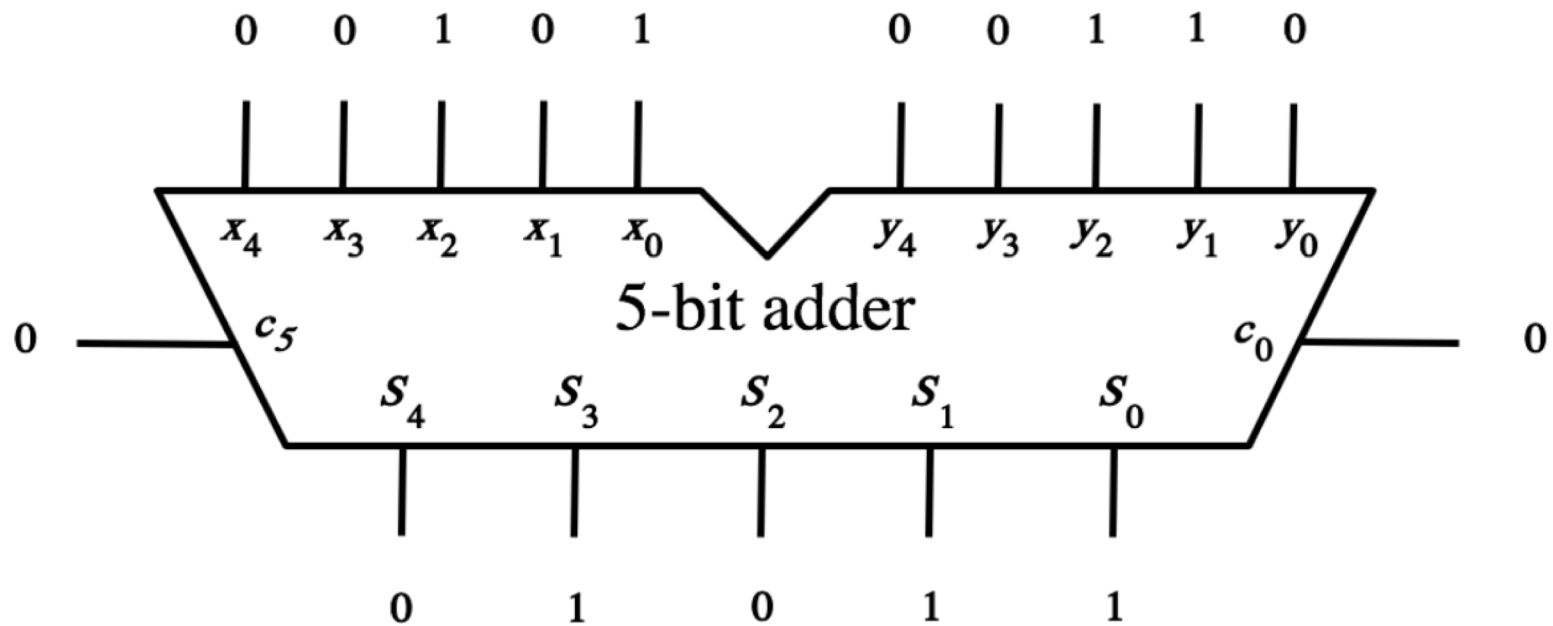


**The x and y lines are typically grouped together for better visualization, but the underlying logic remains the same**

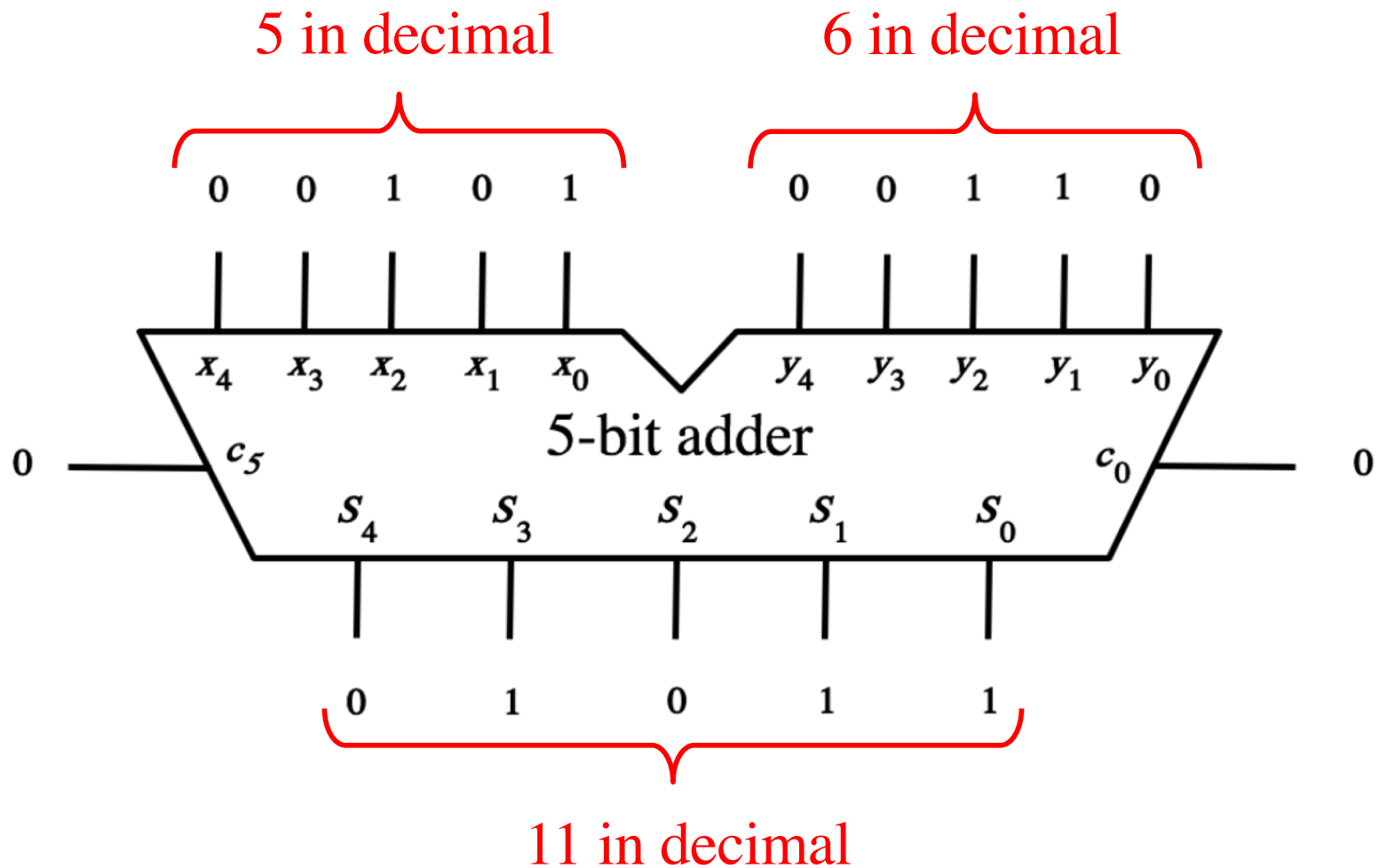


# Example:

## Computing $5+6$ using a 5-bit adder



# Example: Computing 5+6 using a 5-bit adder





# Math Review: Subtraction

$$\begin{array}{r} 39 \\ - 15 \\ \hline ?? \end{array}$$

# Math Review: Subtraction

$$\begin{array}{r} 39 \\ - 15 \\ \hline 24 \end{array}$$



# Math Review: Subtraction

$$\begin{array}{r} \text{—} \quad 82 \\ \quad 61 \\ \hline \quad ?? \end{array}$$

$$\begin{array}{r} \text{—} \quad 48 \\ \quad 26 \\ \hline \quad ?? \end{array}$$

$$\begin{array}{r} \text{—} \quad 32 \\ \quad 11 \\ \hline \quad ?? \end{array}$$

# Math Review: Subtraction

$$\begin{array}{r} 82 \\ - 61 \\ \hline 21 \end{array}$$

$$\begin{array}{r} 48 \\ - 26 \\ \hline 22 \end{array}$$

$$\begin{array}{r} 32 \\ - 11 \\ \hline 21 \end{array}$$



# Math Review: Subtraction

$$\begin{array}{r} \phantom{-} 82 \\ - 64 \\ \hline ?? \end{array}$$

$$\begin{array}{r} \phantom{-} 48 \\ - 29 \\ \hline ?? \end{array}$$

$$\begin{array}{r} \phantom{-} 32 \\ - 13 \\ \hline ?? \end{array}$$

# Math Review: Subtraction

$$\begin{array}{r} \phantom{-} 82 \\ - 64 \\ \hline 18 \end{array}$$

$$\begin{array}{r} \phantom{-} 48 \\ - 29 \\ \hline 19 \end{array}$$

$$\begin{array}{r} \phantom{-} 32 \\ - 13 \\ \hline 19 \end{array}$$



The problems in which row are easier to calculate?

$$\begin{array}{r} \text{—} \quad 82 \\ \quad 61 \\ \hline ?? \end{array}$$

$$\begin{array}{r} \text{—} \quad 48 \\ \quad 26 \\ \hline ?? \end{array}$$

$$\begin{array}{r} \text{—} \quad 32 \\ \quad 11 \\ \hline ?? \end{array}$$

$$\begin{array}{r} \text{—} \quad 82 \\ \quad 64 \\ \hline ?? \end{array}$$

$$\begin{array}{r} \text{—} \quad 48 \\ \quad 29 \\ \hline ?? \end{array}$$

$$\begin{array}{r} \text{—} \quad 32 \\ \quad 13 \\ \hline ?? \end{array}$$

The problems in which row are easier to calculate?

$$\begin{array}{r} \phantom{-} 82 \\ - 61 \\ \hline 21 \end{array}$$

$$\begin{array}{r} \phantom{-} 48 \\ - 26 \\ \hline 22 \end{array}$$

$$\begin{array}{r} \phantom{-} 32 \\ - 11 \\ \hline 21 \end{array}$$

Why?

$$\begin{array}{r} \phantom{-} 82 \\ - 64 \\ \hline 18 \end{array}$$

$$\begin{array}{r} \phantom{-} 48 \\ - 29 \\ \hline 19 \end{array}$$

$$\begin{array}{r} \phantom{-} 32 \\ - 13 \\ \hline 19 \end{array}$$



# Another Way to Do Subtraction

$$82 - 64 = 82 + 100 - 100 - 64$$

## Another Way to Do Subtraction

$$\begin{aligned}82 - 64 &= 82 + 100 - 100 - 64 \\ &= 82 + (100 - 64) - 100\end{aligned}$$

# Another Way to Do Subtraction

$$82 - 64 = 82 + 100 - 100 - 64$$

$$= 82 + (100 - 64) - 100$$

$$= 82 + (99 + 1 - 64) - 100$$

## Another Way to Do Subtraction

$$\begin{aligned}82 - 64 &= 82 + 100 - 100 - 64 \\ &= 82 + (100 - 64) - 100 \\ &= 82 + (99 + 1 - 64) - 100 \\ &= 82 + (99 - 64) + 1 - 100\end{aligned}$$

# Another Way to Do Subtraction

$$82 - 64 = 82 + 100 - 100 - 64$$

$$= 82 + (100 - 64) - 100$$

$$= 82 + (99 + 1 - 64) - 100$$

Does not require borrows

$$= 82 + (99 - 64) + 1 - 100$$

# 9's Complement (subtract each digit from 9)

$$\begin{array}{r} 99 \\ - 64 \\ \hline 35 \end{array}$$

# 10's Complement

(subtract each digit from 9 and add 1 to the result)

$$\begin{array}{r} \phantom{-} 99 \\ - 64 \\ \hline 35 + 1 = 36 \end{array}$$

# Another Way to Do Subtraction

$$82 - 64 = 82 + (99 - 64) + 1 - 100$$



# Another Way to Do Subtraction

9's complement

$$82 - 64 = 82 + (99 - 64) + 1 - 100$$

# Another Way to Do Subtraction

9's complement

$$\begin{aligned}82 - 64 &= 82 + (99 - 64) + 1 - 100 \\ &= 82 + 35 + 1 - 100\end{aligned}$$

# Another Way to Do Subtraction

9's complement

$$82 - 64 = 82 + (99 - 64) + 1 - 100$$

$$= 82 + (35 + 1) - 100$$

10's complement

# Another Way to Do Subtraction

9's complement

$$82 - 64 = 82 + (99 - 64) + 1 - 100$$

$$= 82 + (35 + 1) - 100$$

10's complement

$$= 82 + 36 - 100$$

# Another Way to Do Subtraction

9's complement

$$82 - 64 = 82 + (99 - 64) + 1 - 100$$

10's complement

$$= 82 + (35 + 1) - 100$$

$$= (82 + 36) - 100 \quad // \text{ Add the first two.}$$

$$= 118 - 100$$

# Another Way to Do Subtraction

9's complement

$$82 - 64 = 82 + (99 - 64) + 1 - 100$$

$$= 82 + (35 + 1) - 100$$

10's complement

$$= 82 + 36 - 100 \quad // \text{ Add the first two.}$$

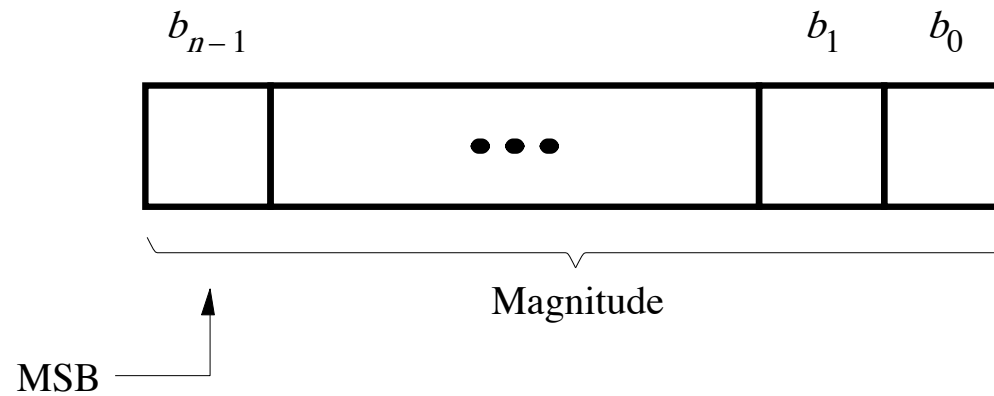
$$= 118 - 100 \quad // \text{ Just delete the leading 1.}$$

// No need to subtract 100.

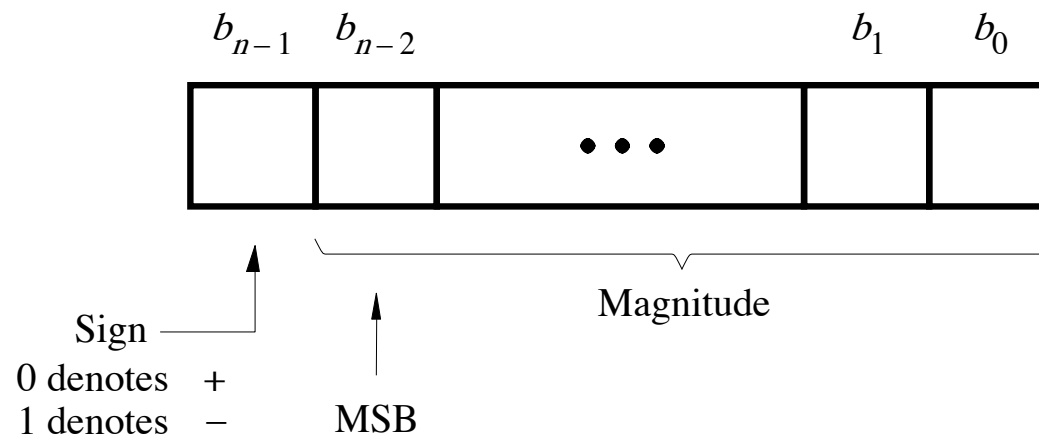
$$= 18$$



# Formats for representation of integers



(a) Unsigned number



(b) Signed number



# Unsigned Representation

$2^7$	$2^6$	$2^5$	$2^4$	$2^3$	$2^2$	$2^1$	$2^0$
0	0	1	0	1	1	0	0

This represents + 44.

# Unsigned Representation

$2^7$	$2^6$	$2^5$	$2^4$	$2^3$	$2^2$	$2^1$	$2^0$
1	0	1	0	1	1	0	0

This represents + 172.

# Three Different Ways to Represent Negative Integer Numbers

- **Sign and magnitude**
- **1's complement**
- **2's complement**

# Three Different Ways to Represent Negative Integer Numbers

- Sign and magnitude
- 1's complement
- 2's complement

only this method is used  
in modern computers

# Interpretation of four-bit signed integers

$b_3b_2b_1b_0$	Sign and magnitude	1's complement	2's complement
0111	+7	+7	+7
0110	+6	+6	+6
0101	+5	+5	+5
0100	+4	+4	+4
0011	+3	+3	+3
0010	+2	+2	+2
0001	+1	+1	+1
0000	+0	+0	+0
1000	-0	-7	-8
1001	-1	-6	-7
1010	-2	-5	-6
1011	-3	-4	-5
1100	-4	-3	-4
1101	-5	-2	-3
1110	-6	-1	-2
1111	-7	-0	-1

# Interpretation of four-bit signed integers

$b_3b_2b_1b_0$	Sign and magnitude	1's complement	2's complement
0111	+7	+7	+7
0110	+6	+6	+6
0101	+5	+5	+5
0100	+4	+4	+4
0011	+3	+3	+3
0010	+2	+2	+2
0001	+1	+1	+1
0000	+0	+0	+0
1000	-0	-7	-8
1001	-1	-6	-7
1010	-2	-5	-6
1011	-3	-4	-5
1100	-4	-3	-4
1101	-5	-2	-3
1110	-6	-1	-2
1111	-7	-0	-1

The top half is the same in all three representations.  
It corresponds to the positive integers.

# Interpretation of four-bit signed integers

$b_3b_2b_1b_0$	Sign and magnitude	1's complement	2's complement
0111	+7	+7	+7
0110	+6	+6	+6
0101	+5	+5	+5
0100	+4	+4	+4
0011	+3	+3	+3
0010	+2	+2	+2
0001	+1	+1	+1
0000	+0	+0	+0
1000	-0	-7	-8
1001	-1	-6	-7
1010	-2	-5	-6
1011	-3	-4	-5
1100	-4	-3	-4
1101	-5	-2	-3
1110	-6	-1	-2
1111	-7	-0	-1

In all three representations the first bit represents the sign.  
If that bit is 1, then the number is negative.

# Interpretation of four-bit signed integers

$b_3b_2b_1b_0$	Sign and magnitude	1's complement	2's complement
0111	+7	+7	+7
0110	+6	+6	+6
0101	+5	+5	+5
0100	+4	+4	+4
0011	+3	+3	+3
0010	+2	+2	+2
0001	+1	+1	+1
0000	+0	+0	+0
1000	-0	-7	-8
1001	-1	-6	-7
1010	-2	-5	-6
1011	-3	-4	-5
1100	-4	-3	-4
1101	-5	-2	-3
1110	-6	-1	-2
1111	-7	-0	-1

Notice that in this representation there are two zeros!



# Interpretation of four-bit signed integers

$b_3b_2b_1b_0$	Sign and magnitude	1's complement	2's complement
0111	+7	+7	+7
0110	+6	+6	+6
0101	+5	+5	+5
0100	+4	+4	+4
0011	+3	+3	+3
0010	+2	+2	+2
0001	+1	+1	+1
0000	+0	+0	+0
1000	-0	-7	-8
1001	-1	-6	-7
1010	-2	-5	-6
1011	-3	-4	-5
1100	-4	-3	-4
1101	-5	-2	-3
1110	-6	-1	-2
1111	-7	-0	-1

There are two zeros in this representation as well!

# Interpretation of four-bit signed integers

$b_3b_2b_1b_0$	Sign and magnitude	1's complement	2's complement
0111	+7	+7	+7
0110	+6	+6	+6
0101	+5	+5	+5
0100	+4	+4	+4
0011	+3	+3	+3
0010	+2	+2	+2
0001	+1	+1	+1
0000	+0	+0	+0
1000	-0	-7	-8
1001	-1	-6	-7
1010	-2	-5	-6
1011	-3	-4	-5
1100	-4	-3	-4
1101	-5	-2	-3
1110	-6	-1	-2
1111	-7	-0	-1

In this representation there is one more negative number.

# **Sign and Magnitude**

# Sign and Magnitude Representation (using the left-most bit as the sign)

sign	$2^6$	$2^5$	$2^4$	$2^3$	$2^2$	$2^1$	$2^0$
0	0	1	0	1	1	0	0

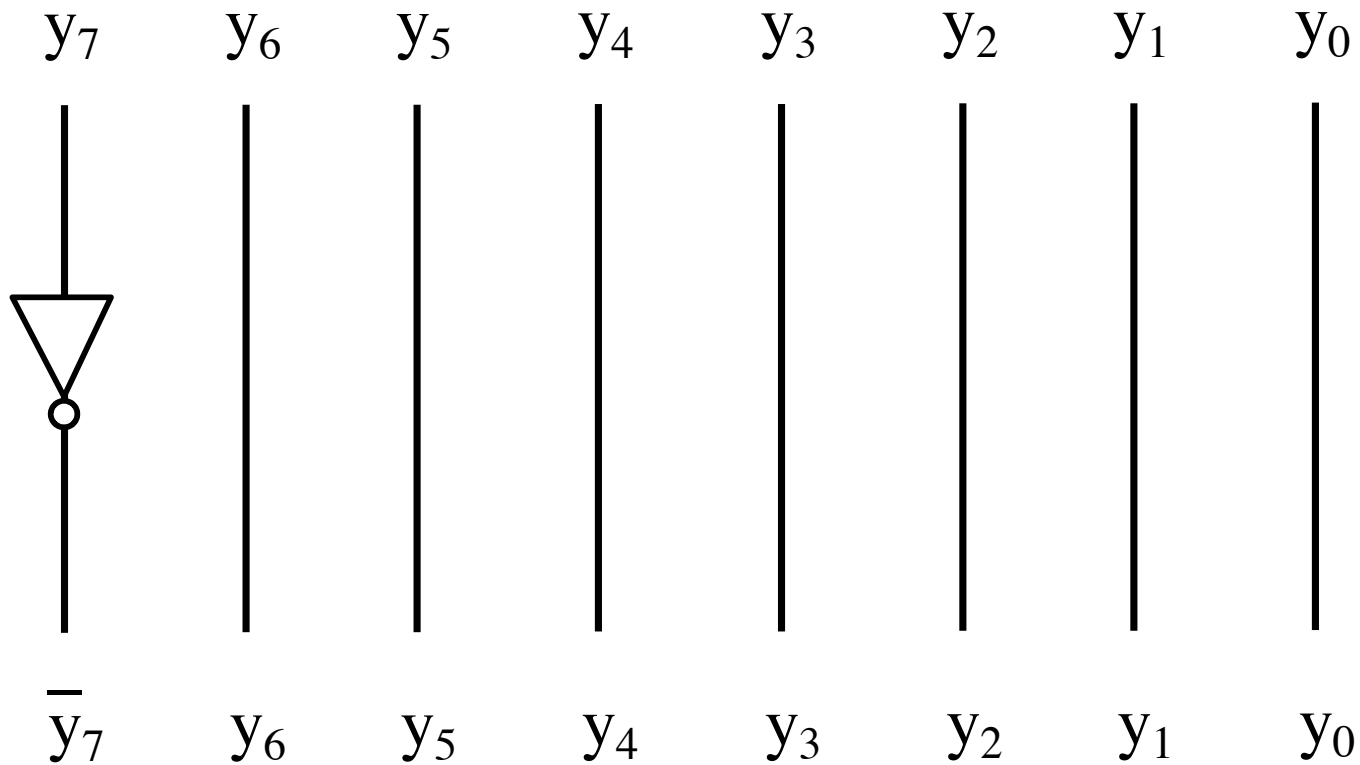
This represents + 44.

# Sign and Magnitude Representation (using the left-most bit as the sign)

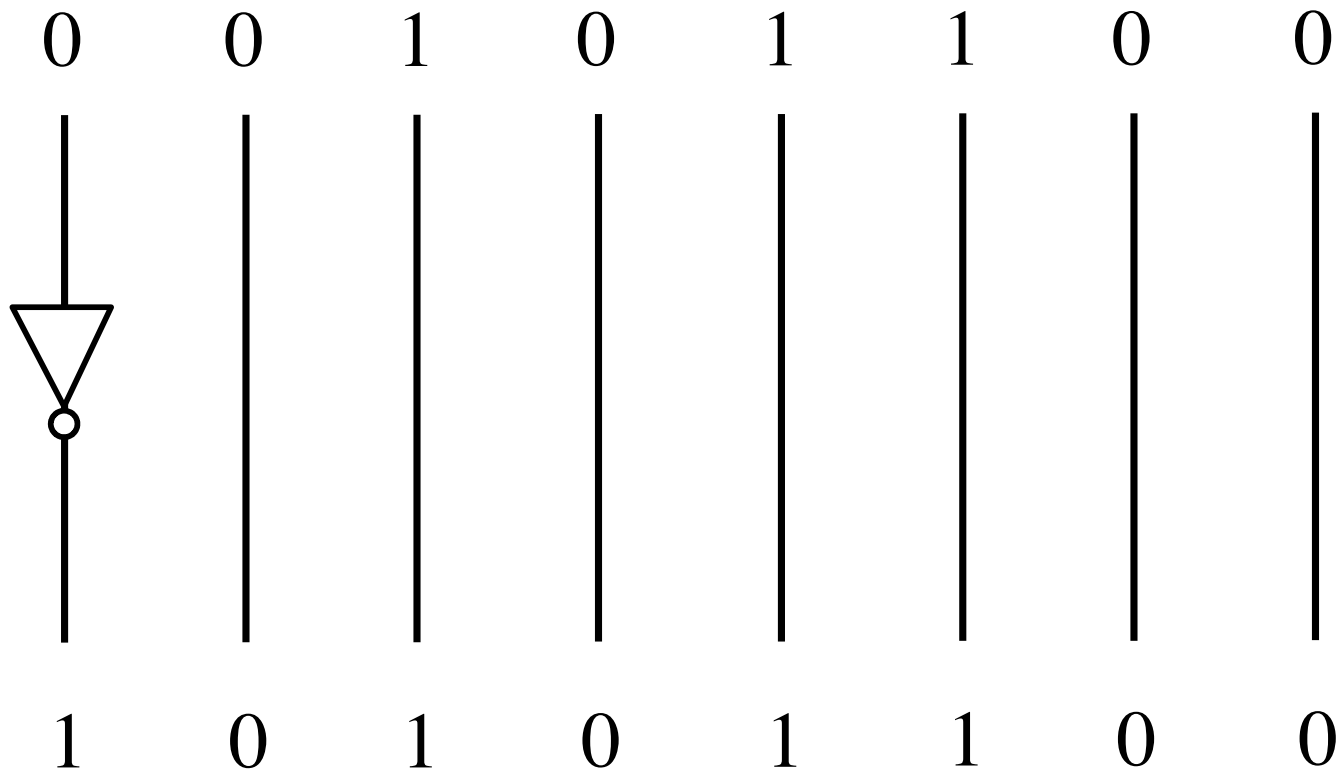
sign	$2^6$	$2^5$	$2^4$	$2^3$	$2^2$	$2^1$	$2^0$
1	0	1	0	1	1	0	0

This represents  $-44$ .

# Circuit for negating a number stored in sign and magnitude representation



# Circuit for negating a number stored in sign and magnitude representation



# **1's Complement**



# **1' s complement (subtract each digit from 1)**

Let  $K$  be the negative equivalent of an  $n$ -bit positive number  $P$ .

Then, in 1' s complement representation  $K$  is obtained by subtracting  $P$  from  $2^n - 1$ , namely

$$K = (2^n - 1) - P$$

This means that  $K$  can be obtained by inverting all bits of  $P$ .

# 1' s complement (subtract each digit from 1)

Let  $K$  be the negative equivalent of an 8-bit positive number  $P$ .

Then, in 1' s complement representation  $K$  is obtained by subtracting  $P$  from  $2^8 - 1$ , namely

$$K = (2^8 - 1) - P = 255 - P$$

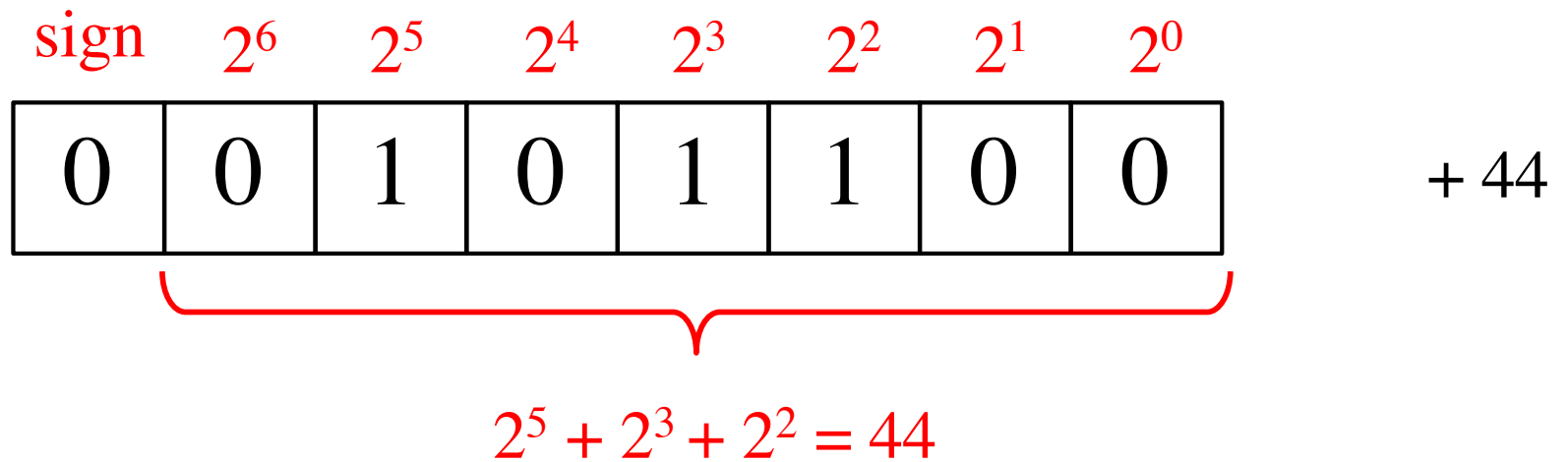
This means that  $K$  can be obtained by inverting all bits of  $P$ .

Provided that  $P$  is between 0 and 127, because the most significant bit must be zero to indicate that it is positive.

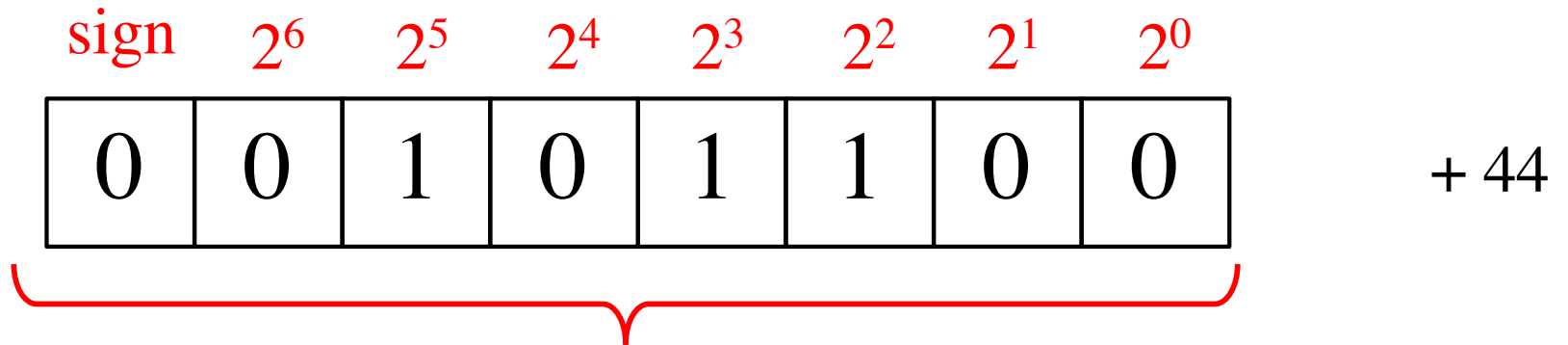
# 1's Complement Representation

sign	$2^6$	$2^5$	$2^4$	$2^3$	$2^2$	$2^1$	$2^0$
0	0	1	0	1	1	0	0

# 1's Complement Representation



# 1's Complement Representation



+ 44 in 1's complement representation

# 1's Complement Representation

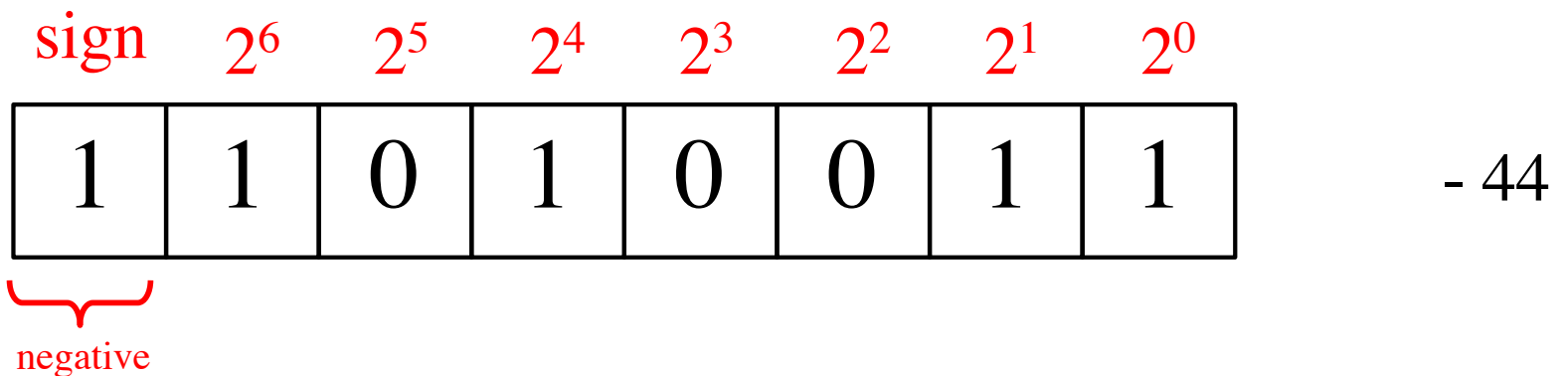
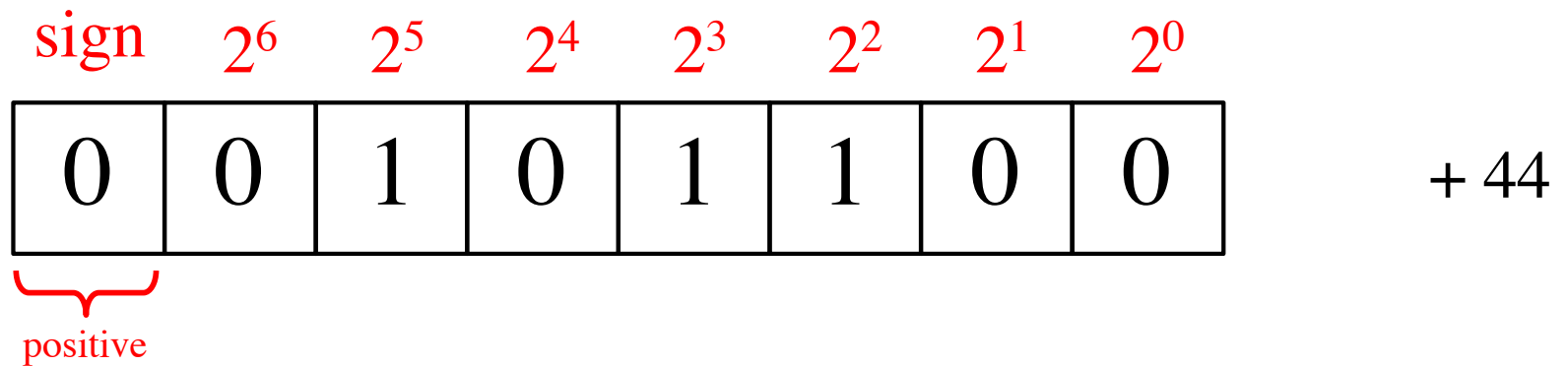
(invert all the bits to negate the number)

sign	$2^6$	$2^5$	$2^4$	$2^3$	$2^2$	$2^1$	$2^0$	
0	0	1	0	1	1	0	0	+ 44

sign	$2^6$	$2^5$	$2^4$	$2^3$	$2^2$	$2^1$	$2^0$	
1	1	0	1	0	0	1	1	- 44

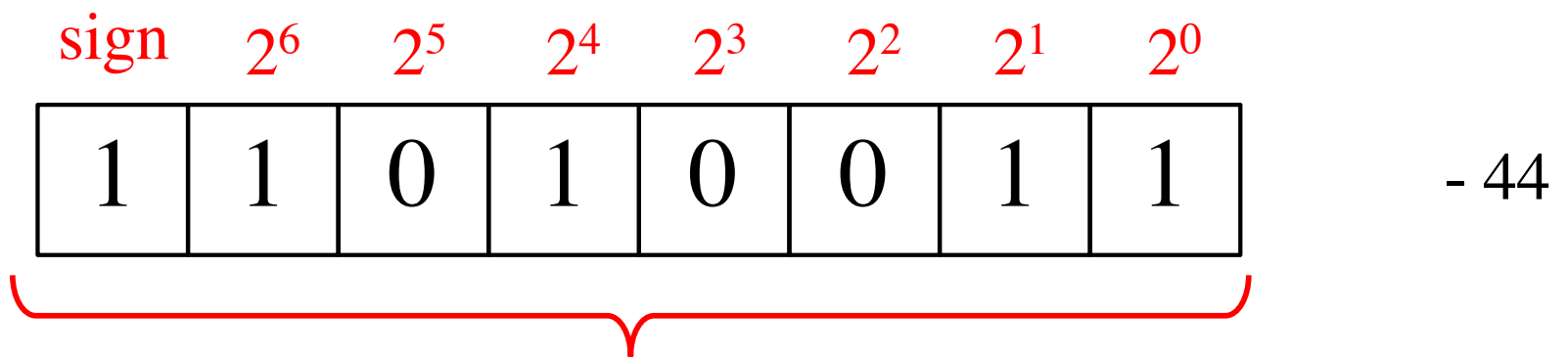
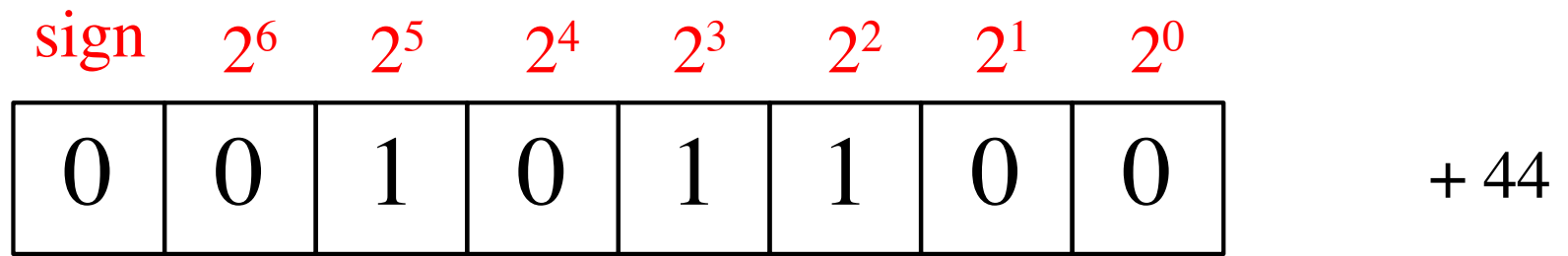
# 1's Complement Representation

(invert all the bits to negate the number)



# 1's Complement Representation

(invert all the bits to negate the number)

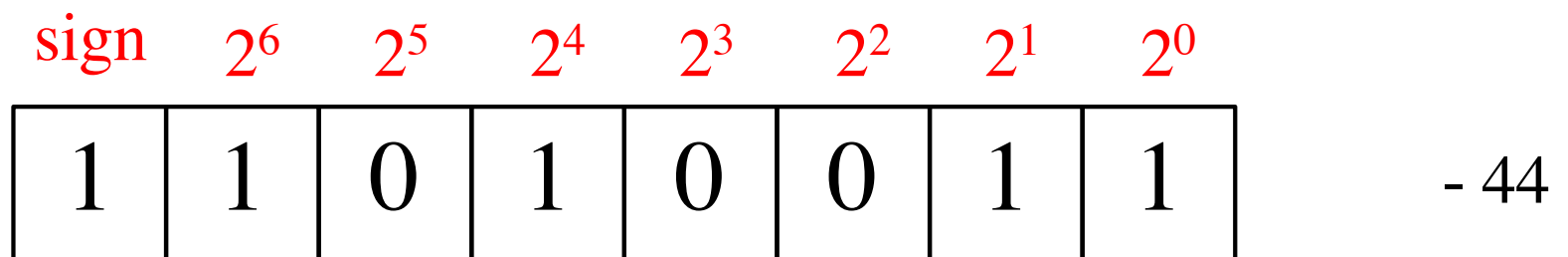
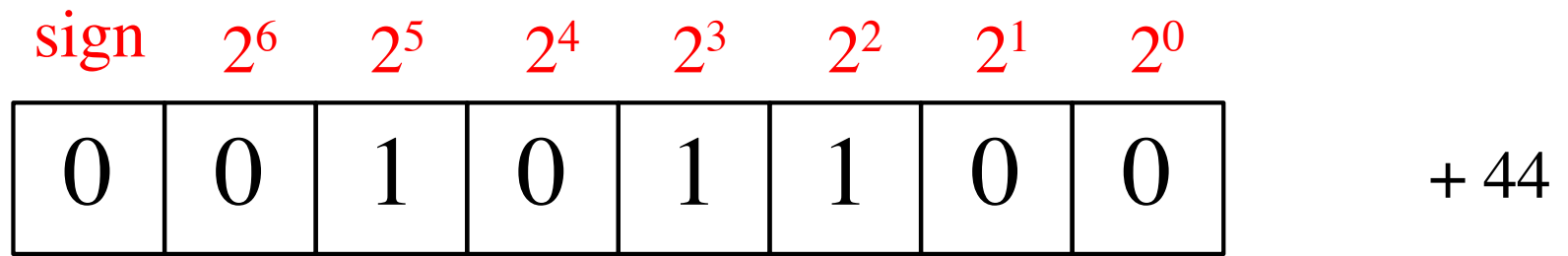


$$2^7 + 2^6 + 2^4 + 2^1 + 2^0 = 211 \text{ (as unsigned)}$$



# 1's Complement Representation

(invert all the bits to negate the number)



$$211 = 255 - 44 \text{ (as unsigned)}$$

# 1's Complement Representation

(invert all the bits to negate the number)

sign	$2^6$	$2^5$	$2^4$	$2^3$	$2^2$	$2^1$	$2^0$	
0	0	1	0	1	1	0	0	+ 44

sign	$2^6$	$2^5$	$2^4$	$2^3$	$2^2$	$2^1$	$2^0$	
1	1	0	1	0	0	1	1	- 44



- 44 in 1's complement representation

# 1's complement (subtract each digit from 1)

$$\begin{array}{r} \phantom{-} 1 \phantom{0} 1 \phantom{0} 1 \phantom{0} 1 \phantom{0} 1 \phantom{0} 1 \phantom{0} 1 \\ - 0 \phantom{0} 0 \phantom{0} 1 \phantom{0} 0 \phantom{0} 1 \phantom{0} 1 \phantom{0} 0 \phantom{0} 0 \\ \hline 1 \phantom{0} 1 \phantom{0} 0 \phantom{0} 1 \phantom{0} 0 \phantom{0} 0 \phantom{0} 1 \phantom{0} 1 \end{array}$$

# 1's complement (subtract each digit from 1)

No need to borrow!

$$\begin{array}{r} \phantom{-} 1 \phantom{0} 1 \phantom{0} 1 \phantom{0} 1 \phantom{0} 1 \phantom{0} 1 \phantom{0} 1 \\ - 0 \phantom{0} 0 \phantom{0} 1 \phantom{0} 0 \phantom{0} 1 \phantom{0} 1 \phantom{0} 0 \phantom{0} 0 \\ \hline 1 \phantom{0} 1 \phantom{0} 0 \phantom{0} 1 \phantom{0} 0 \phantom{0} 0 \phantom{0} 1 \phantom{0} 1 \end{array}$$

# 1's complement (subtract each digit from 1)

	1	1	1	1	1	1	1	1	255
-	0	0	1	0	1	1	0	0	44
<hr/>									
	1	1	0	1	0	0	1	1	

# 1's complement (subtract each digit from 1)

— 1 1 1 1 1 1 1 1 1  
— 0 0 1 0 1 1 0 0

---

1 1 0 1 0 0 1 1 211

$$211 = 255 - 44 \text{ (as unsigned)}$$

# 1's complement (subtract each digit from 1)

$$\begin{array}{r} \phantom{-} 1 \phantom{0} 1 \phantom{0} 1 \phantom{0} 1 \phantom{0} 1 \phantom{0} 1 \phantom{0} 1 \\ - 0 \phantom{0} 0 \phantom{0} 1 \phantom{0} 0 \phantom{0} 1 \phantom{0} 1 \phantom{0} 0 \phantom{0} 0 \end{array}$$

---

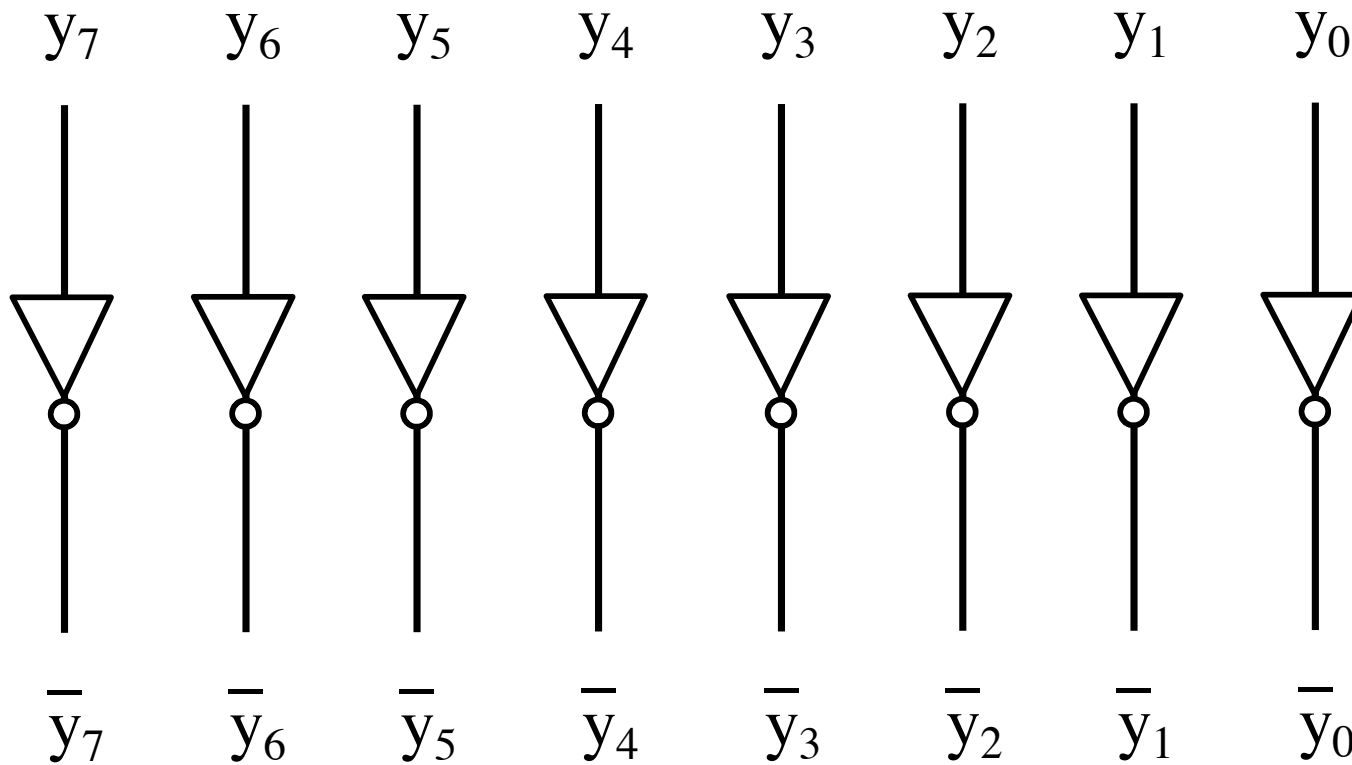
1	1	0	1	0	0	1	1	- 44
---	---	---	---	---	---	---	---	------

$211 = 255 - 44$  (as unsigned)

or

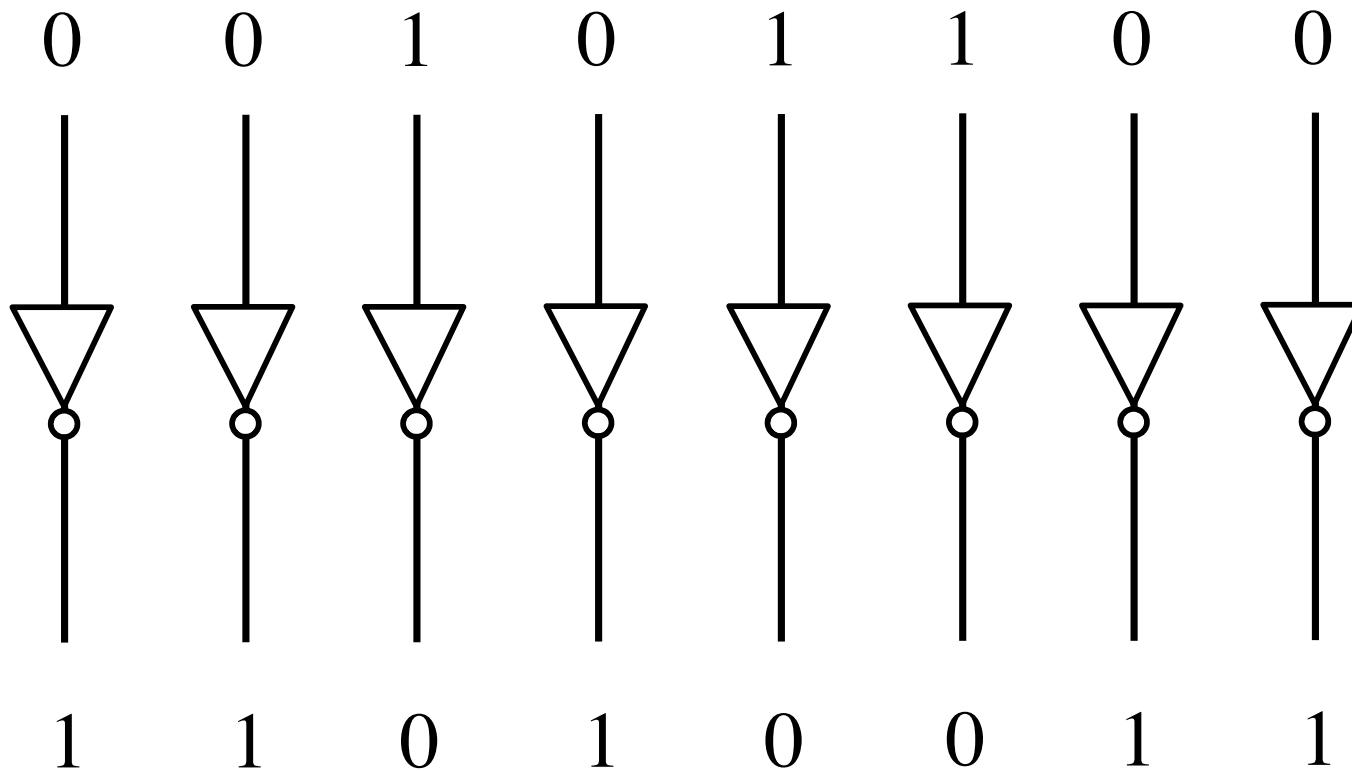
- 44 in 1's complement representation

# Circuit for negating a number stored in 1's complement representation



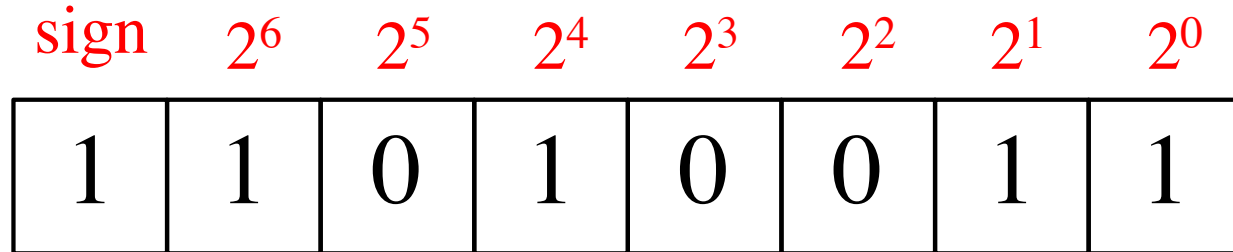


# Circuit for negating a number stored in 1's complement representation



**This works in reverse too  
(from negative to positive)**

# 1's Complement Representation



- 44

# 1's Complement Representation

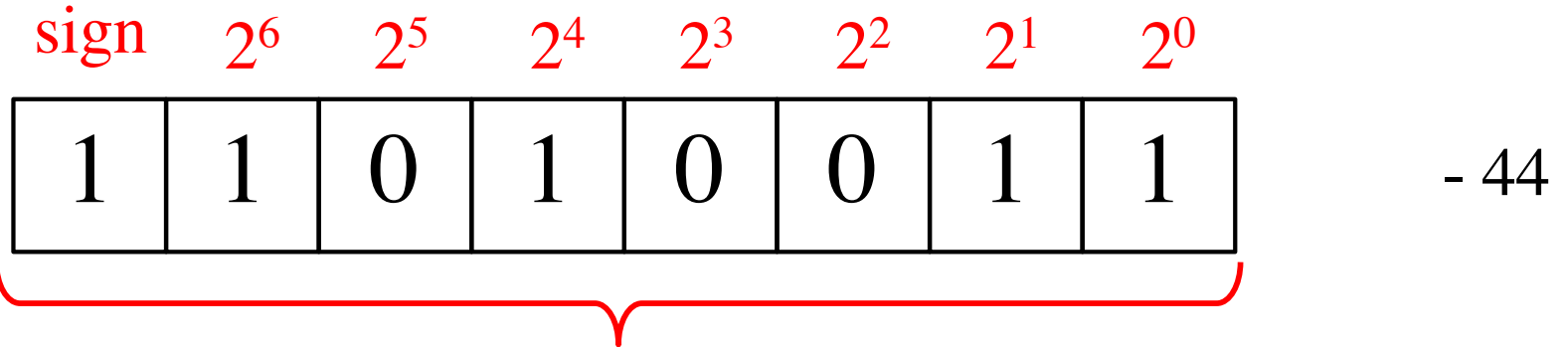
(invert all the bits to negate the number)

sign	$2^6$	$2^5$	$2^4$	$2^3$	$2^2$	$2^1$	$2^0$	
1	1	0	1	0	0	1	1	- 44

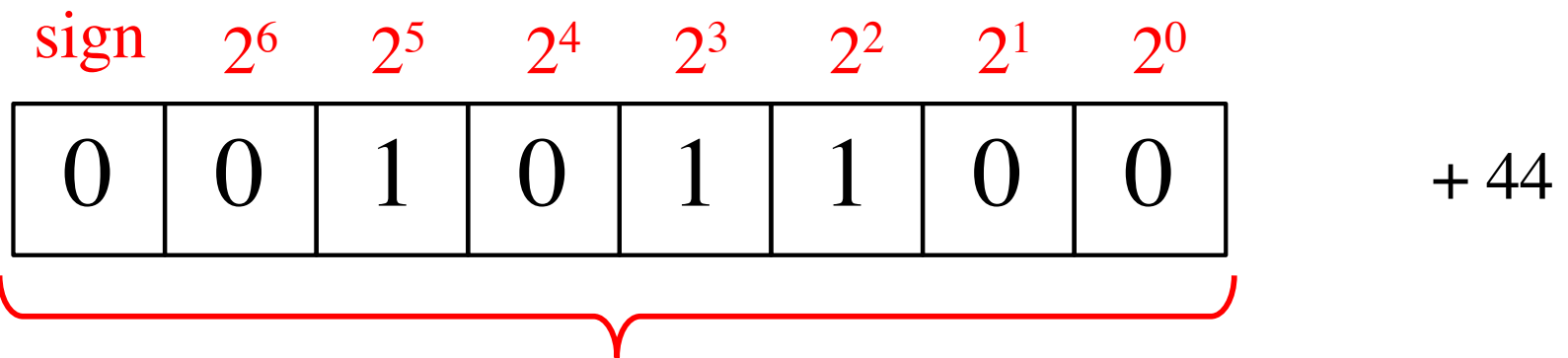
sign	$2^6$	$2^5$	$2^4$	$2^3$	$2^2$	$2^1$	$2^0$	
0	0	1	0	1	1	0	0	+ 44

# 1's Complement Representation

(invert all the bits to negate the number)



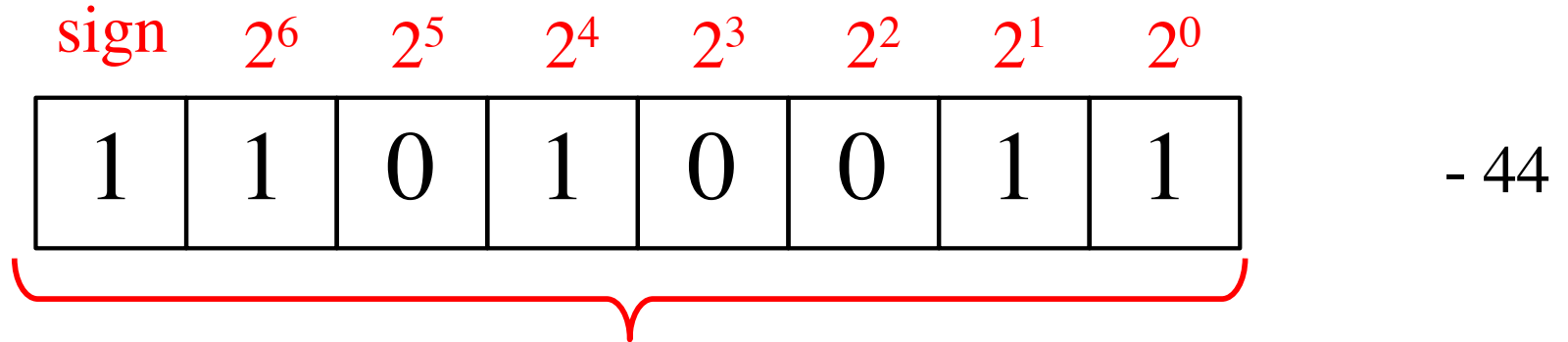
211 (as unsigned)



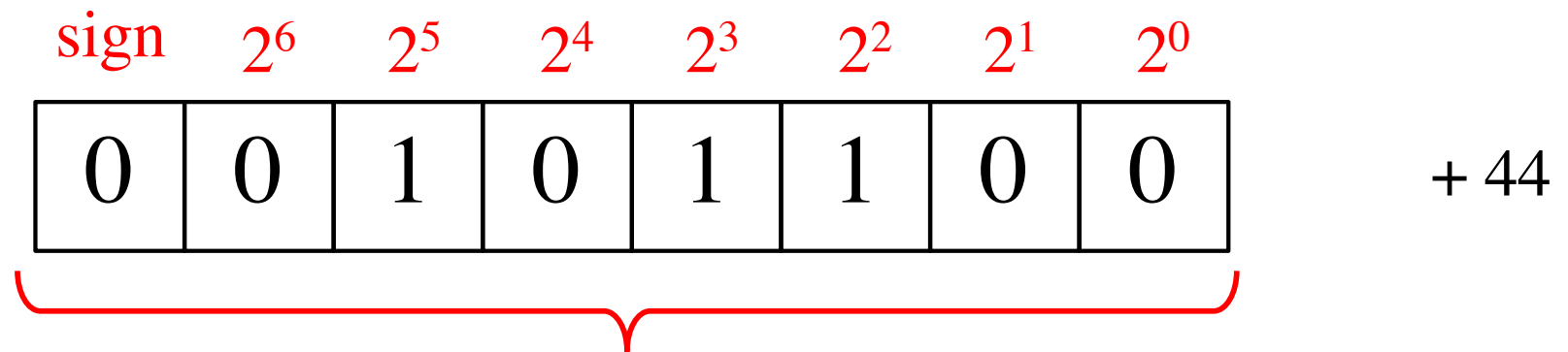
$44 = 255 - 211$  (as unsigned)

# 1's Complement Representation

(invert all the bits to negate the number)



- 44 in 1's complement representation



+ 44 in 1's complement representation

**Find the 1's complement of ...**

0 1 0 1

0 0 1 0

0 0 1 1

0 1 1 1

# Find the 1's complement of ...

0 1 0 1

1 0 1 0

0 0 1 0

1 1 0 1

0 0 1 1

1 1 0 0

0 1 1 1

1 0 0 0

Just flip 1's to 0's and vice versa.



# A) Example of 1's complement addition

$$\begin{array}{r}
 (+5) \quad 0101 \\
 +(+2) \quad +0010 \\
 \hline
 (+7) \quad 0111
 \end{array}$$

$b_3b_2b_1b_0$	1's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	-7
1001	-6
1010	-5
1011	-4
1100	-3
1101	-2
1110	-1
1111	-0

[ Figure 3.8 from the textbook ]

# A) Example of 1's complement addition

$$\begin{array}{r}
 (+5) \\
 +(+2) \\
 \hline
 (+7)
 \end{array}
 \qquad
 \begin{array}{r}
 \color{red}{0101} \\
 + \color{green}{0010} \\
 \hline
 \color{blue}{0111}
 \end{array}$$

$b_3b_2b_1b_0$	1's complement
<span style="background-color: #add8e6;">0111</span>	<span style="background-color: #add8e6;">+7</span>
0110	+6
<span style="background-color: #ff6347;">0101</span>	<span style="background-color: #ff6347;">+5</span>
0100	+4
0011	+3
<span style="background-color: #38a83d;">0010</span>	<span style="background-color: #38a83d;">+2</span>
0001	+1
0000	+0
1000	-7
1001	-6
1010	-5
1011	-4
1100	-3
1101	-2
1110	-1
1111	-0

## B) Example of 1's complement addition

$$\begin{array}{r}
 (-5) \\
 +(+2) \\
 \hline
 (-3)
 \end{array}
 \qquad
 \begin{array}{r}
 1010 \\
 +0010 \\
 \hline
 1100
 \end{array}$$

$b_3b_2b_1b_0$	1's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	-7
1001	-6
1010	-5
1011	-4
1100	-3
1101	-2
1110	-1
1111	-0

## B) Example of 1's complement addition

$$\begin{array}{r}
 (-5) \\
 +(+2) \\
 \hline
 (-3)
 \end{array}
 \qquad
 \begin{array}{r}
 \color{red}{1010} \\
 + \color{green}{0010} \\
 \hline
 \color{blue}{1100}
 \end{array}$$

$b_3b_2b_1b_0$	1's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	-7
1001	-6
1010	-5
1011	-4
1100	-3
1101	-2
1110	-1
1111	-0

# C) Example of 1's complement addition

$$\begin{array}{r}
 (+5) \quad 0101 \\
 +(-2) \quad +1101 \\
 \hline
 (+3) \quad 10010
 \end{array}$$

$b_3b_2b_1b_0$	1's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	-7
1001	-6
1010	-5
1011	-4
1100	-3
1101	-2
1110	-1
1111	-0

[ Figure 3.8 from the textbook ]

# C) Example of 1's complement addition

$$\begin{array}{r}
 (+5) \\
 +(-2) \\
 \hline
 (+3)
 \end{array}
 \quad
 \begin{array}{r}
 0101 \\
 +1101 \\
 \hline
 10010
 \end{array}$$

$b_3b_2b_1b_0$	1's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	-7
1001	-6
1010	-5
1011	-4
1100	-3
1101	-2
1110	-1
1111	-0

# C) Example of 1's complement addition

$$\begin{array}{r}
 (+5) \\
 +(-2) \\
 \hline
 (+3)
 \end{array}
 \quad
 \begin{array}{r}
 0101 \\
 +1101 \\
 \hline
 10010
 \end{array}$$

But this is 2!

$b_3b_2b_1b_0$	1's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	-7
1001	-6
1010	-5
1011	-4
1100	-3
1101	-2
1110	-1
1111	-0

# C) Example of 1's complement addition

$$\begin{array}{r}
 (+5) \quad 0101 \\
 +(-2) \quad +1101 \\
 \hline
 (+3) \quad 10010 \\
 \hline
 \quad \quad \quad 1 \\
 \quad \quad \quad \leftarrow \\
 \quad \quad \quad \hline
 \quad \quad \quad 0011
 \end{array}$$

We need to perform one more addition to get the result.

$b_3b_2b_1b_0$	1's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	-7
1001	-6
1010	-5
1011	-4
1100	-3
1101	-2
1110	-1
1111	-0



# C) Example of 1's complement addition

$$\begin{array}{r}
 (+5) \quad 0101 \\
 +(-2) \quad +1101 \\
 \hline
 (+3) \quad 10010 \\
 \hline
 \end{array}$$

We need to perform one more addition to get the result.

$b_3b_2b_1b_0$	1's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	-7
1001	-6
1010	-5
1011	-4
1100	-3
1101	-2
1110	-1
1111	-0

# D) Example of 1's complement addition

$$\begin{array}{r}
 (-5) \\
 + (-2) \\
 \hline
 (-7)
 \end{array}
 \qquad
 \begin{array}{r}
 1010 \\
 + 1101 \\
 \hline
 10111
 \end{array}$$

$b_3b_2b_1b_0$	1's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	-7
1001	-6
1010	-5
1011	-4
1100	-3
1101	-2
1110	-1
1111	-0

[ Figure 3.8 from the textbook ]

# D) Example of 1's complement addition

$$\begin{array}{r}
 (-5) \\
 + (-2) \\
 \hline
 (-7)
 \end{array}
 \quad
 \begin{array}{r}
 1010 \\
 + 1101 \\
 \hline
 10111
 \end{array}$$

$b_3b_2b_1b_0$	1's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	-7
1001	-6
1010	-5
1011	-4
1100	-3
1101	-2
1110	-1
1111	-0

# D) Example of 1's complement addition

$$\begin{array}{r}
 (-5) \\
 + (-2) \\
 \hline
 (-7)
 \end{array}
 \quad
 \begin{array}{r}
 + \begin{array}{r} 1010 \\ 1101 \\ \hline 10111 \end{array}
 \end{array}$$

But this is +7!

$b_3b_2b_1b_0$	1's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	-7
1001	-6
1010	-5
1011	-4
1100	-3
1101	-2
1110	-1
1111	-0

# D) Example of 1's complement addition

$$\begin{array}{r}
 (-5) \\
 + (-2) \\
 \hline
 (-7)
 \end{array}
 \qquad
 \begin{array}{r}
 1010 \\
 + 1101 \\
 \hline
 10111 \\
 \begin{array}{l} \color{blue}{\lrcorner} \\ \color{blue}{\rightarrow} \end{array} 1 \\
 \hline
 1000
 \end{array}$$

We need to perform one more addition to get the result.

$b_3b_2b_1b_0$	1's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	-7
1001	-6
1010	-5
1011	-4
1100	-3
1101	-2
1110	-1
1111	-0

# D) Example of 1's complement addition

$$\begin{array}{r}
 (-5) \\
 + (-2) \\
 \hline
 (-7)
 \end{array}
 \qquad
 \begin{array}{r}
 1010 \\
 + 1101 \\
 \hline
 10111 \\
 \text{⌋} \quad \text{⌋} \quad \text{⌋} \quad \text{⌋} \quad \text{⌋} \\
 \text{⌋} \quad \text{⌋} \quad \text{⌋} \quad \text{⌋} \quad \text{⌋} \\
 \hline
 1000
 \end{array}$$

We need to perform one more addition to get the result.

$b_3b_2b_1b_0$	1's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	-7
1001	-6
1010	-5
1011	-4
1100	-3
1101	-2
1110	-1
1111	-0

# **2's Complement**

# **2' s complement**

**(subtract each digit from 1 and add 1 to the result)**

Let  $K$  be the negative equivalent of an  $n$ -bit positive number  $P$ .

Then, in 2' s complement representation  $K$  is obtained by subtracting  $P$  from  $2^n$ , namely

$$K = 2^n - P$$



# 2' s complement

(subtract each digit from 1 and add 1 to the result)

Let K be the negative equivalent of an 8-bit positive number P.

Then, in 2' s complement representation K is obtained by subtracting P from  $2^8$ , namely

$$K = 2^8 - P = 256 - P$$

# 2's Complement Representation

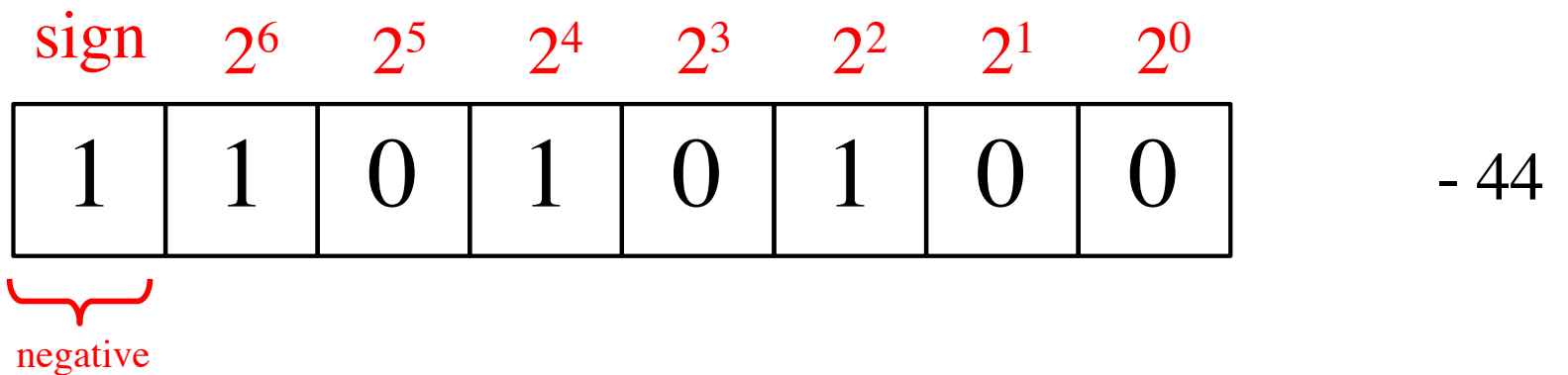
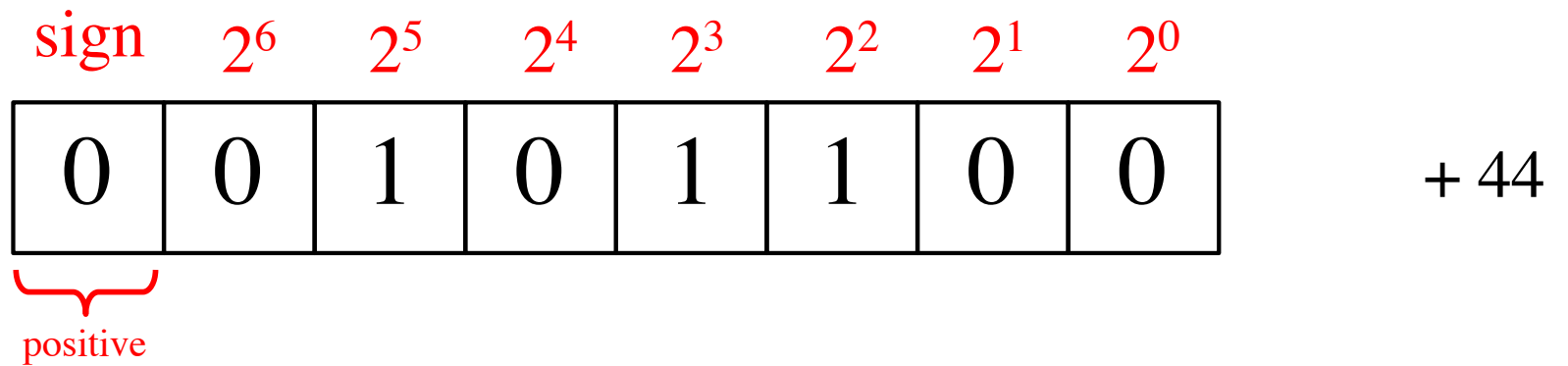
sign	$2^6$	$2^5$	$2^4$	$2^3$	$2^2$	$2^1$	$2^0$	
0	0	1	0	1	1	0	0	+ 44

# 2's Complement Representation

sign	$2^6$	$2^5$	$2^4$	$2^3$	$2^2$	$2^1$	$2^0$	
0	0	1	0	1	1	0	0	+ 44

sign	$2^6$	$2^5$	$2^4$	$2^3$	$2^2$	$2^1$	$2^0$	
1	1	0	1	0	1	0	0	- 44

# 2's Complement Representation



# 2's Complement Representation

sign	$2^6$	$2^5$	$2^4$	$2^3$	$2^2$	$2^1$	$2^0$	
0	0	1	0	1	1	0	0	+ 44

sign	$2^6$	$2^5$	$2^4$	$2^3$	$2^2$	$2^1$	$2^0$	
1	1	0	1	0	1	0	0	- 44



$$212 = 256 - 44$$

# Deriving 2' s complement

For a positive n-bit number P, let  $K_1$  and  $K_2$  denote its 1' s and 2' s complements, respectively.

$$K_1 = (2^n - 1) - P$$

$$K_2 = 2^n - P$$

Since  $K_2 = K_1 + 1$ , it is evident that in a logic circuit the 2' s complement can be computed by inverting all bits of P and then adding 1 to the resulting 1' s-complement number.

# Deriving 2' s complement

For a positive 8-bit number P, let  $K_1$  and  $K_2$  denote its 1' s and 2' s complements, respectively.

$$K_1 = (2^n - 1) - P = 255 - P$$

$$K_2 = 2^n - P = 256 - P$$

Since  $K_2 = K_1 + 1$ , it is evident that in a logic circuit the 2' s complement can be computed by inverting all bits of P and then adding 1 to the resulting 1' s-complement number.

**Find the 2' s complement of ...**

0 1 0 1

0 0 1 0

0 1 0 0

0 1 1 1



## Find the 2' s complement of ...

0 1 0 1

1 0 1 0

0 0 1 0

1 1 0 1

0 1 0 0

1 0 1 1

0 1 1 1

1 0 0 0

Invert all bits.

# Find the 2's complement of ...

$$\begin{array}{r} 0101 \\ + 1010 \\ + \phantom{00}1 \\ \hline 1011 \end{array}$$

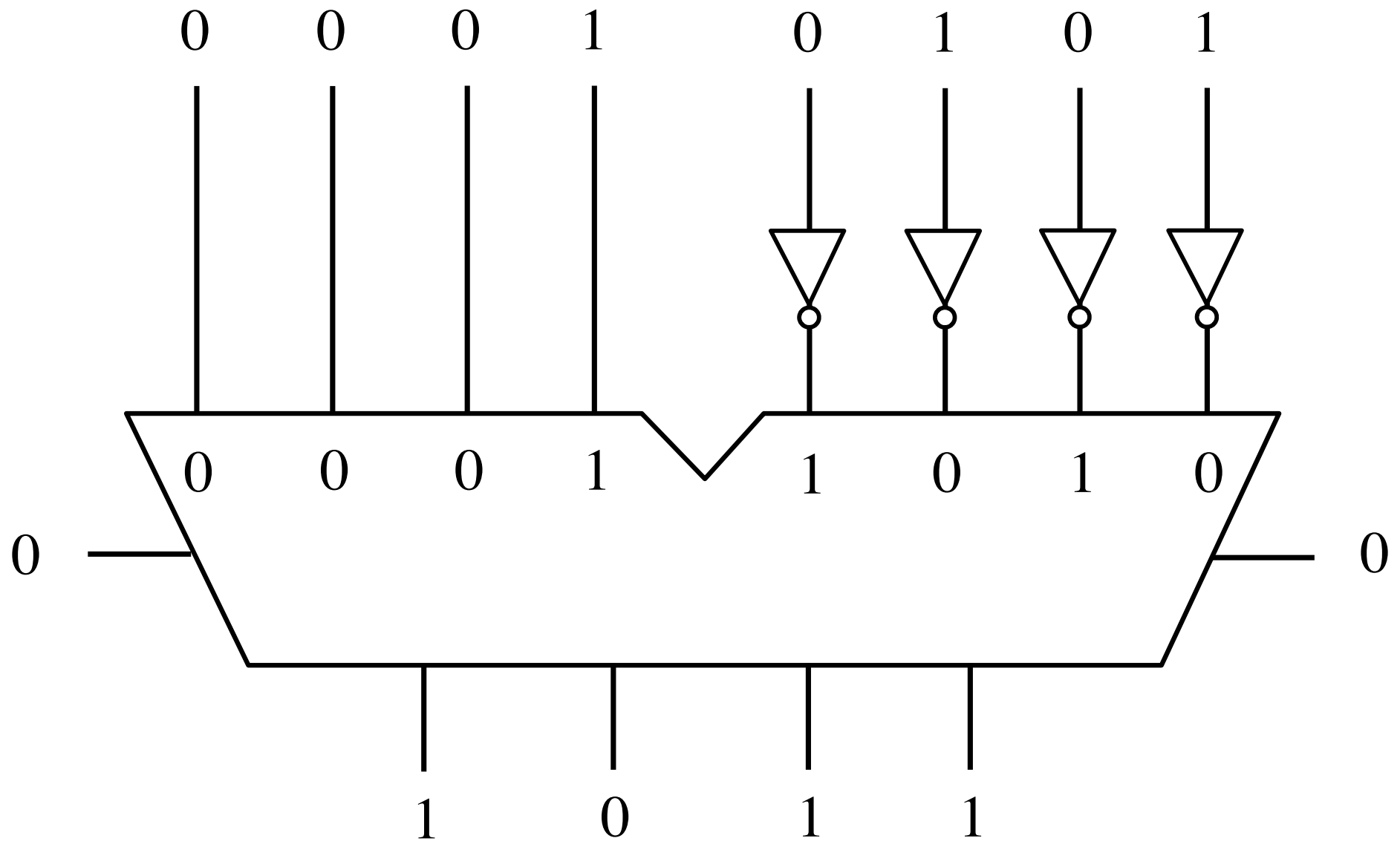
$$\begin{array}{r} 0010 \\ + 1101 \\ + \phantom{00}1 \\ \hline 1110 \end{array}$$

$$\begin{array}{r} 0100 \\ + 1011 \\ + \phantom{00}1 \\ \hline 1100 \end{array}$$

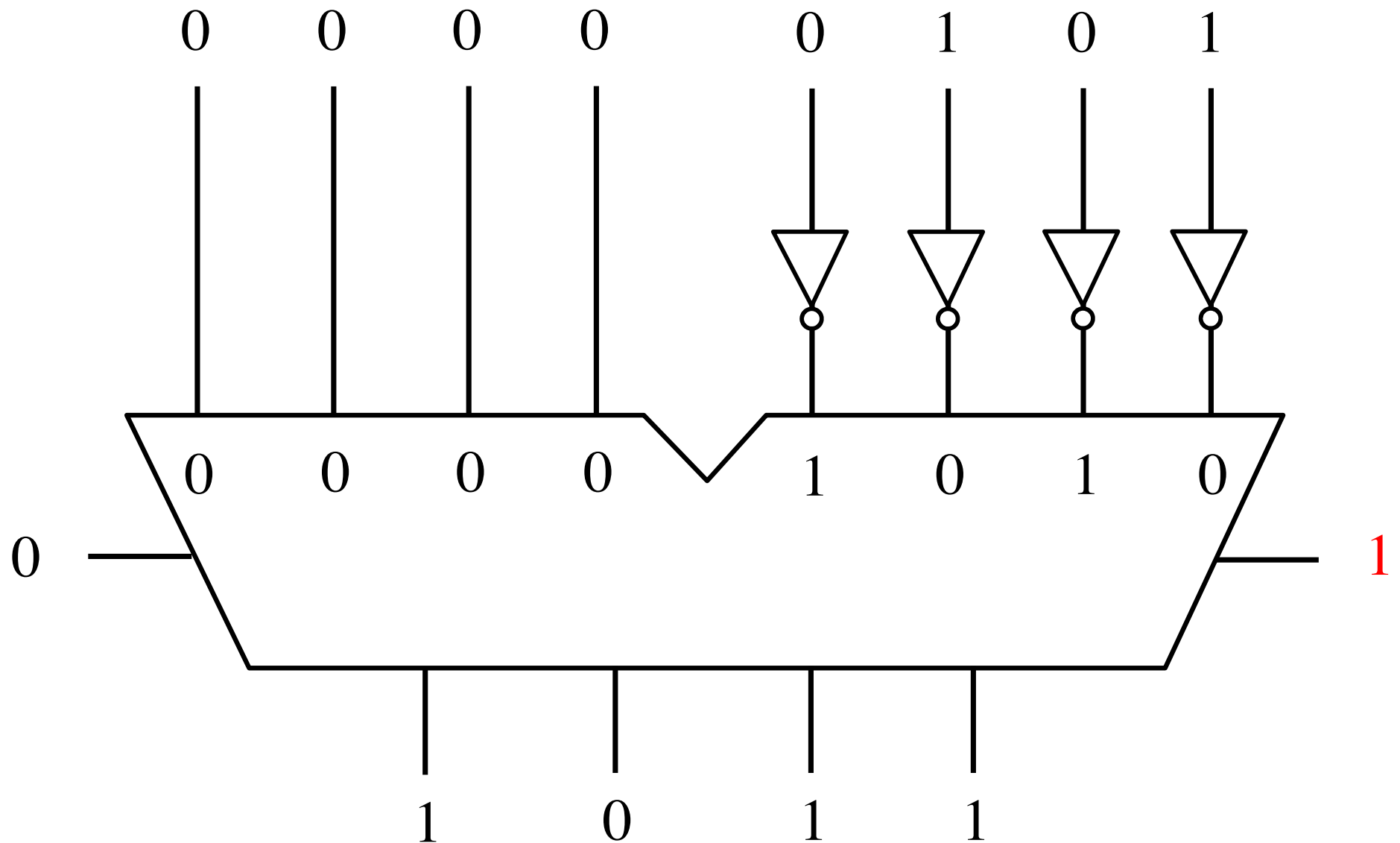
$$\begin{array}{r} 0111 \\ + 1000 \\ + \phantom{00}1 \\ \hline 1001 \end{array}$$

Then add 1.

# Circuit #1 for negating a number stored in 2's complement representation



# Circuit #2 for negating a number stored in 2's complement representation



# Quick way (**for a human**) to find 2's complement

- **Scan the binary number from right to left**
- **Copy all bits that are 0 from right to left**
- **Stop at the first 1**
- **Copy that 1 as well**
- **Invert all remaining bits**

**Find the 2' s complement of ...**

0 1 0 1

0 0 1 0

0 1 0 0

0 1 1 1

# Find the 2's complement of ...

0 1 0 1

. . . .

0 0 1 0

. . . 0

0 1 0 0

. . 0 0

0 1 1 1

. . . .

Copy all bits that are 0 from right to left.

# Find the 2's complement of ...

0 1 0 1  
. . . 1

0 0 1 0  
. . 1 0

0 1 0 0  
. 1 0 0

0 1 1 1  
. . . 1

Stop at the first 1. Copy that 1 as well.



# Find the 2's complement of ...

0 1 0 1

1 0 1 1

0 0 1 0

1 1 1 0

0 1 0 0

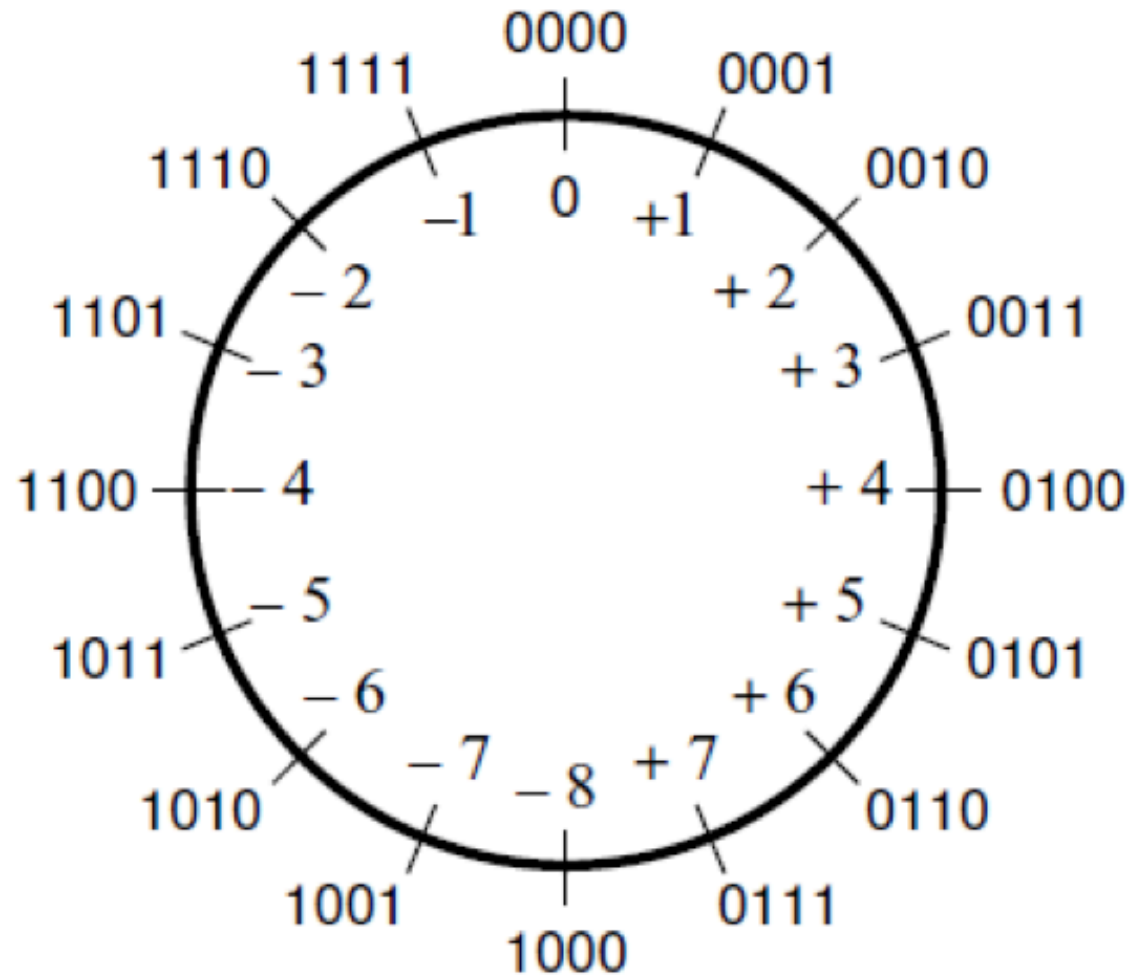
1 1 0 0

0 1 1 1

1 0 0 1

Invert all remaining bits.

# The number circle for 2's complement



[ Figure 3.11a from the textbook ]

# A) Example of 2's complement addition

$$\begin{array}{r}
 (+5) \\
 + (+2) \\
 \hline
 (+7)
 \end{array}
 \qquad
 \begin{array}{r}
 0101 \\
 + 0010 \\
 \hline
 0111
 \end{array}$$

$b_3b_2b_1b_0$	2's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	-8
1001	-7
1010	-6
1011	-5
1100	-4
1101	-3
1110	-2
1111	-1


## B) Example of 2's complement addition

$$\begin{array}{r}
 (-5) \\
 + (+2) \\
 \hline
 (-3)
 \end{array}
 \qquad
 \begin{array}{r}
 1011 \\
 + 0010 \\
 \hline
 1101
 \end{array}$$

$b_3b_2b_1b_0$	2's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	-8
1001	-7
1010	-6
1011	-5
1100	-4
1101	-3
1110	-2
1111	-1

# C) Example of 2's complement addition


$$\begin{array}{r}
 (+5) \quad \quad \quad 0101 \\
 + (-2) \quad \quad \quad 1110 \\
 \hline
 (+3) \quad \quad \quad 10011
 \end{array}$$


  
 ignore

$b_3b_2b_1b_0$	2's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	-8
1001	-7
1010	-6
1011	-5
1100	-4
1101	-3
1110	-2
1111	-1

# D) Example of 2's complement addition

$$\begin{array}{r}
 (-5) \\
 + (-2) \\
 \hline
 (-7)
 \end{array}
 \qquad
 \begin{array}{r}
 1011 \\
 + 1110 \\
 \hline
 11001
 \end{array}$$


  
 ignore

$b_3b_2b_1b_0$	2's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	-8
1001	-7
1010	-6
1011	-5
1100	-4
1101	-3
1110	-2
1111	-1



# **Naming Ambiguity: 2's Complement**

**2's complement has two different meanings:**

- **representation for signed integer numbers**
- **algorithm for computing the 2's complement (regardless of the representation of the number)**



# Naming Ambiguity: 2's Complement

2's complement has two different meanings:

- representation for signed integer numbers  
**in 2's complement**
- algorithm for computing the 2's complement  
(regardless of the representation of the number)  
**take the 2's complement (or negate)**

# Example of 2' s complement subtraction

$$\begin{array}{r} (+5) \\ - (+2) \\ \hline (+3) \end{array} \quad \begin{array}{r} 0101 \\ - 0010 \\ \hline \end{array} \quad \Rightarrow \quad \begin{array}{r} 0101 \\ + 1110 \\ \hline 10011 \\ \uparrow \\ \text{ignore} \end{array}$$

$\Rightarrow$  means take the 2's complement (or negate)

# Example of 2's complement subtraction

$$\begin{array}{r} (+5) \\ \textcircled{-} (+2) \\ \hline (+3) \end{array} \quad \begin{array}{r} 0101 \\ - 0010 \\ \hline \end{array} \quad \Rightarrow \quad \begin{array}{r} 0101 \\ \textcircled{+} 1110 \\ \hline 10011 \\ \uparrow \\ \text{ignore} \end{array}$$

Notice that the minus changes to a plus.

$\Rightarrow$  means take the 2's complement (or negate)

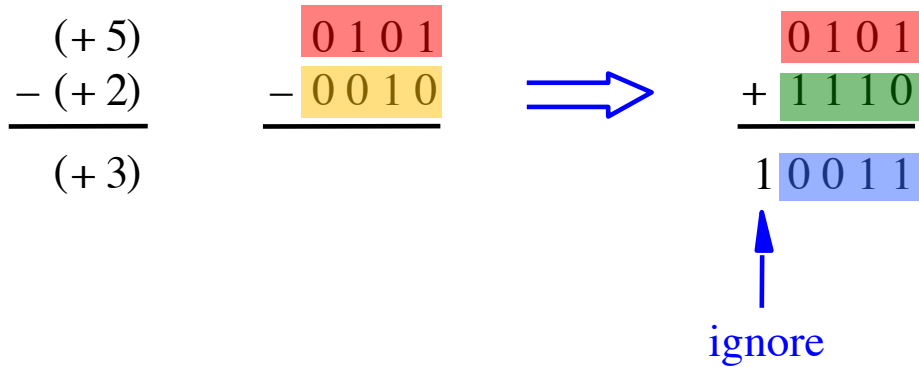
# Example of 2's complement subtraction

$$\begin{array}{r}
 (+5) \\
 - (+2) \\
 \hline
 (+3)
 \end{array}
 \quad
 \begin{array}{r}
 \color{red}{0101} \\
 - \color{yellow}{0010} \\
 \hline
 \end{array}
 \Rightarrow
 \begin{array}{r}
 \phantom{0}0101 \\
 + 1110 \\
 \hline
 10011 \\
 \uparrow \\
 \text{ignore}
 \end{array}$$

$b_3b_2b_1b_0$	2's complement
0111	+7
0110	+6
<b>0101</b>	<b>+5</b>
0100	+4
0011	+3
<b>0010</b>	<b>+2</b>
0001	+1
0000	+0
1000	-8
1001	-7
1010	-6
1011	-5
1100	-4
1101	-3
1110	-2
1111	-1

[ Figure 3.10 from the textbook ]

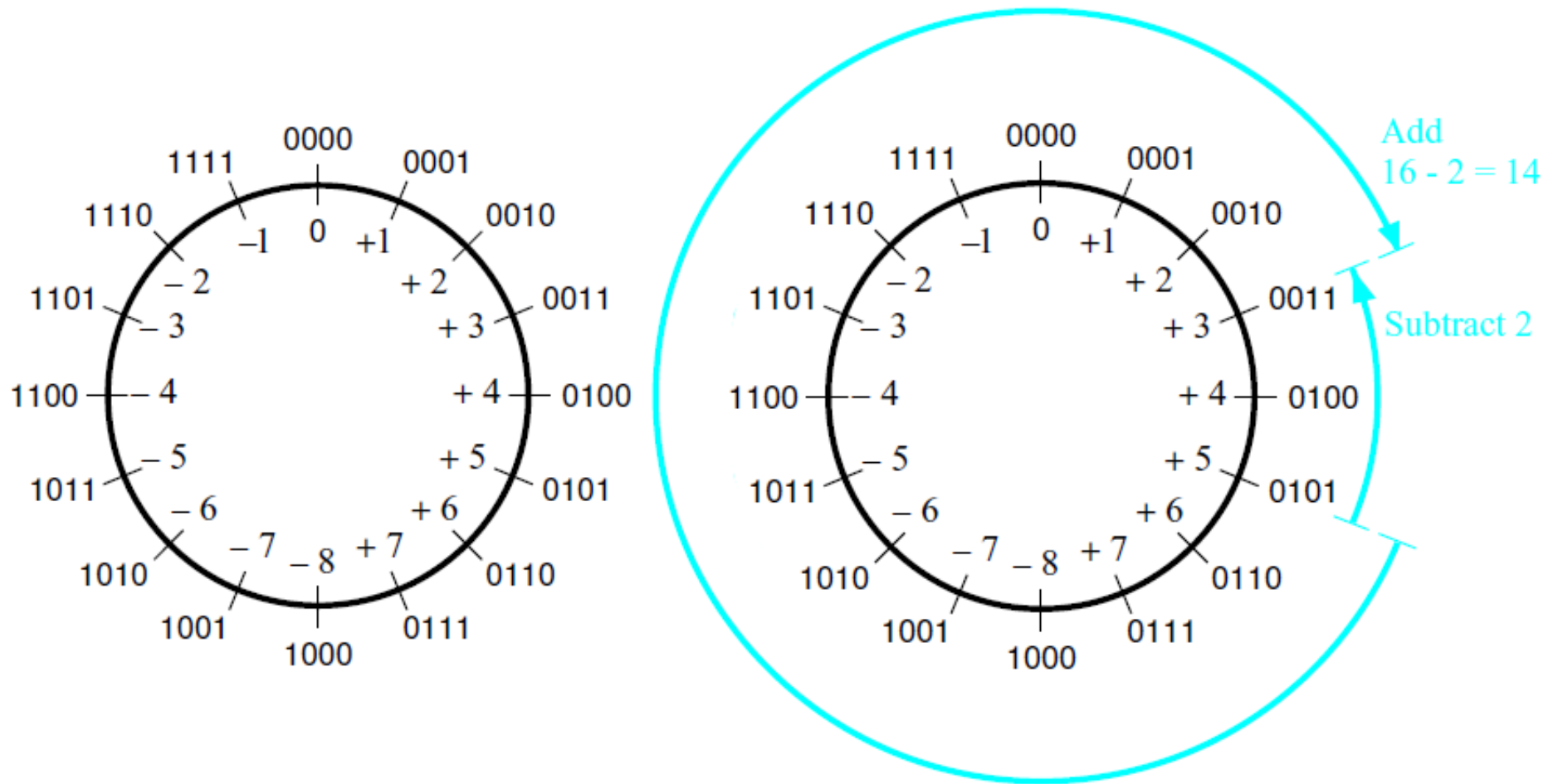
# Example of 2's complement subtraction



$b_3b_2b_1b_0$	2's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	-8
1001	-7
1010	-6
1011	-5
1100	-4
1101	-3
1110	-2
1111	-1

[ Figure 3.10 from the textbook ]

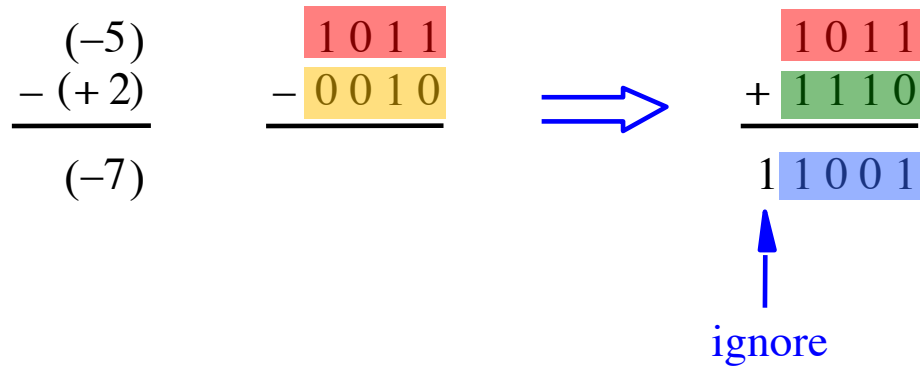
# Graphical interpretation of four-bit 2's complement numbers



(a) The number circle

(b) Subtracting 2 by adding its 2's complement

# Example of 2's complement subtraction



$b_3b_2b_1b_0$	2's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	-8
1001	-7
1010	-6
1011	-5
1100	-4
1101	-3
1110	-2
1111	-1

[ Figure 3.10 from the textbook ]

# Example of 2's complement subtraction

$$\begin{array}{r}
 (+5) \\
 - (-2) \\
 \hline
 (+7)
 \end{array}
 \quad
 \begin{array}{r}
 \boxed{0101} \\
 - \boxed{1110} \\
 \hline
 \end{array}
 \Rightarrow
 \begin{array}{r}
 \boxed{0101} \\
 + \boxed{0010} \\
 \hline
 \boxed{0111}
 \end{array}$$

$b_3b_2b_1b_0$	2's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	-8
1001	-7
1010	-6
1011	-5
1100	-4
1101	-3
1110	-2
1111	-1



# Example of 2's complement subtraction

$$\begin{array}{r}
 (-5) \\
 - (-2) \\
 \hline
 (-3)
 \end{array}
 \quad
 \begin{array}{r}
 \color{red}{1011} \\
 - \color{yellow}{1110} \\
 \hline
 \end{array}
 \quad
 \Rightarrow
 \quad
 \begin{array}{r}
 \color{red}{1011} \\
 + \color{green}{0010} \\
 \hline
 \color{blue}{1101}
 \end{array}$$

$b_3b_2b_1b_0$	2's complement
0111	+7
0110	+6
0101	+5
0100	+4
0011	+3
0010	+2
0001	+1
0000	+0
1000	-8
1001	-7
1010	-6
1011	-5
1100	-4
1101	-3
1110	-2
1111	-1

[ Figure 3.10 from the textbook ]

# Taking the 2's complement negates the number

decimal	b <sub>3</sub> b <sub>2</sub> b <sub>1</sub> b <sub>0</sub>	take the 2's complement	b <sub>3</sub> b <sub>2</sub> b <sub>1</sub> b <sub>0</sub>	decimal
+7	0111	⇒	1001	-7
+6	0110	⇒	1010	-6
+5	0101	⇒	1011	-5
+4	0100	⇒	1100	-4
+3	0011	⇒	1101	-3
+2	0010	⇒	1110	-2
+1	0001	⇒	1111	-1
+0	0000	⇒	0000	+0
-8	1000	⇒	1000	-8
-7	1001	⇒	0111	+7
-6	1010	⇒	0110	+6
-5	1011	⇒	0101	+5
-4	1100	⇒	0100	+4
-3	1101	⇒	0011	+3
-2	1110	⇒	0010	+2
-1	1111	⇒	0001	+1

# Taking the 2's complement negates the number

decimal	b <sub>3</sub> b <sub>2</sub> b <sub>1</sub> b <sub>0</sub>	take the 2's complement	b <sub>3</sub> b <sub>2</sub> b <sub>1</sub> b <sub>0</sub>	decimal
+7	0111	⇒	1001	-7
+6	0110	⇒	1010	-6
+5	0101	⇒	1011	-5
+4	0100	⇒	1100	-4
+3	0011	⇒	1101	-3
+2	0010	⇒	1110	-2
+1	0001	⇒	1111	-1
+0	0000	⇒	0000	+0
-8	1000	⇒	1000	-8
-7	1001	⇒	0111	+7
-6	1010	⇒	0110	+6
-5	1011	⇒	0101	+5
-4	1100	⇒	0100	+4
-3	1101	⇒	0011	+3
-2	1110	⇒	0010	+2
-1	1111	⇒	0001	+1

This is the only exception

# Taking the 2's complement negates the number

decimal	b <sub>3</sub> b <sub>2</sub> b <sub>1</sub> b <sub>0</sub>	take the 2's complement	b <sub>3</sub> b <sub>2</sub> b <sub>1</sub> b <sub>0</sub>	decimal
+7	0111	⇒	1001	-7
+6	0110	⇒	1010	-6
+5	0101	⇒	1011	-5
+4	0100	⇒	1100	-4
+3	0011	⇒	1101	-3
+2	0010	⇒	1110	-2
+1	0001	⇒	1111	-1
+0	0000	⇒	0000	+0
-8	1000	⇒	1000	-8
-7	1001	⇒	0111	+7
-6	1010	⇒	0110	+6
-5	1011	⇒	0101	+5
-4	1100	⇒	0100	+4
-3	1101	⇒	0011	+3
-2	1110	⇒	0010	+2
-1	1111	⇒	0001	+1

And this one too.

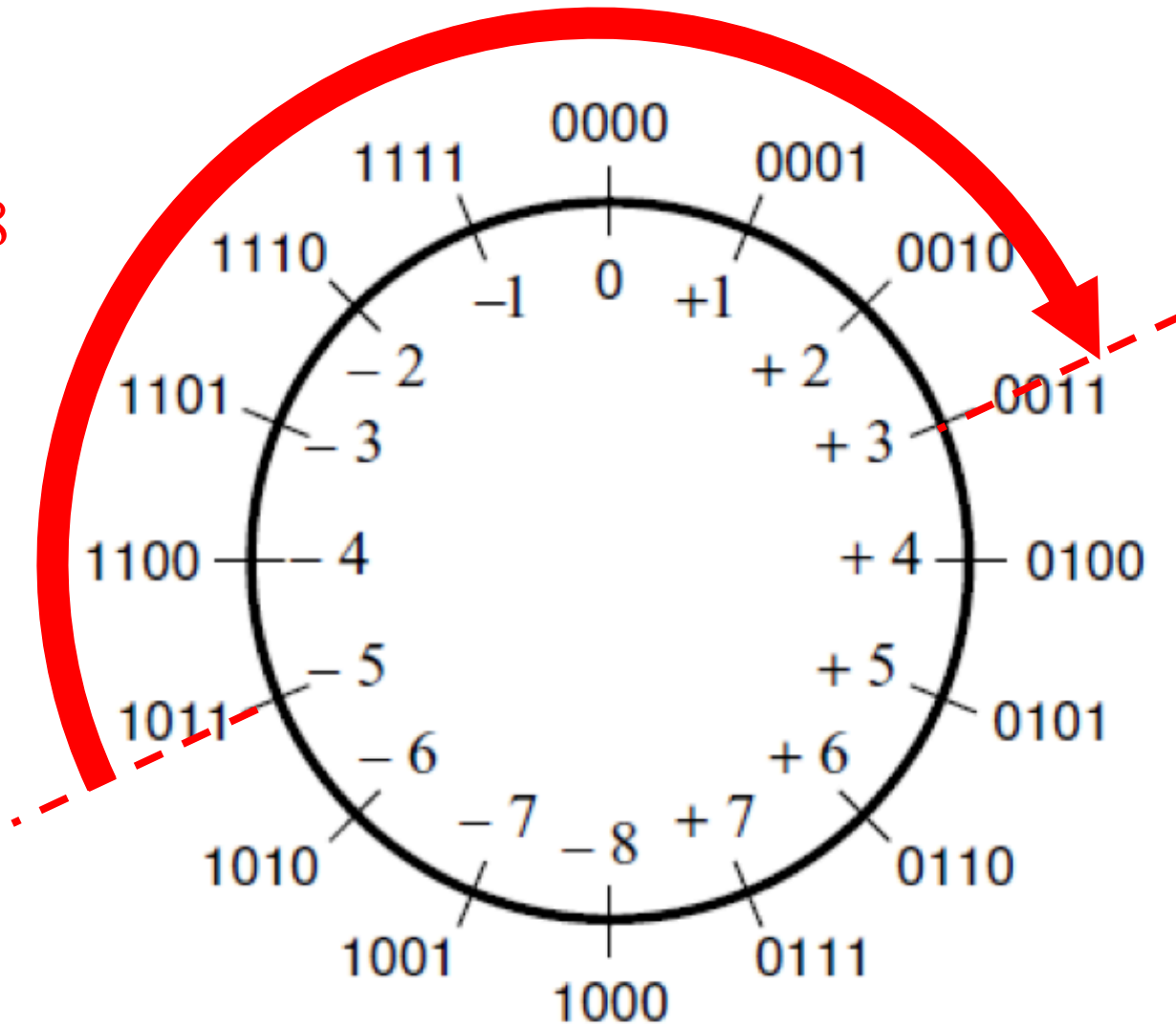
# But that exception does not matter

$$\begin{array}{r} (-5) \\ - (-8) \\ \hline (+3) \end{array} \quad \begin{array}{r} 1011 \\ - 1000 \\ \hline \end{array} \quad \Rightarrow \quad \begin{array}{r} 1011 \\ + 1000 \\ \hline 10011 \end{array}$$

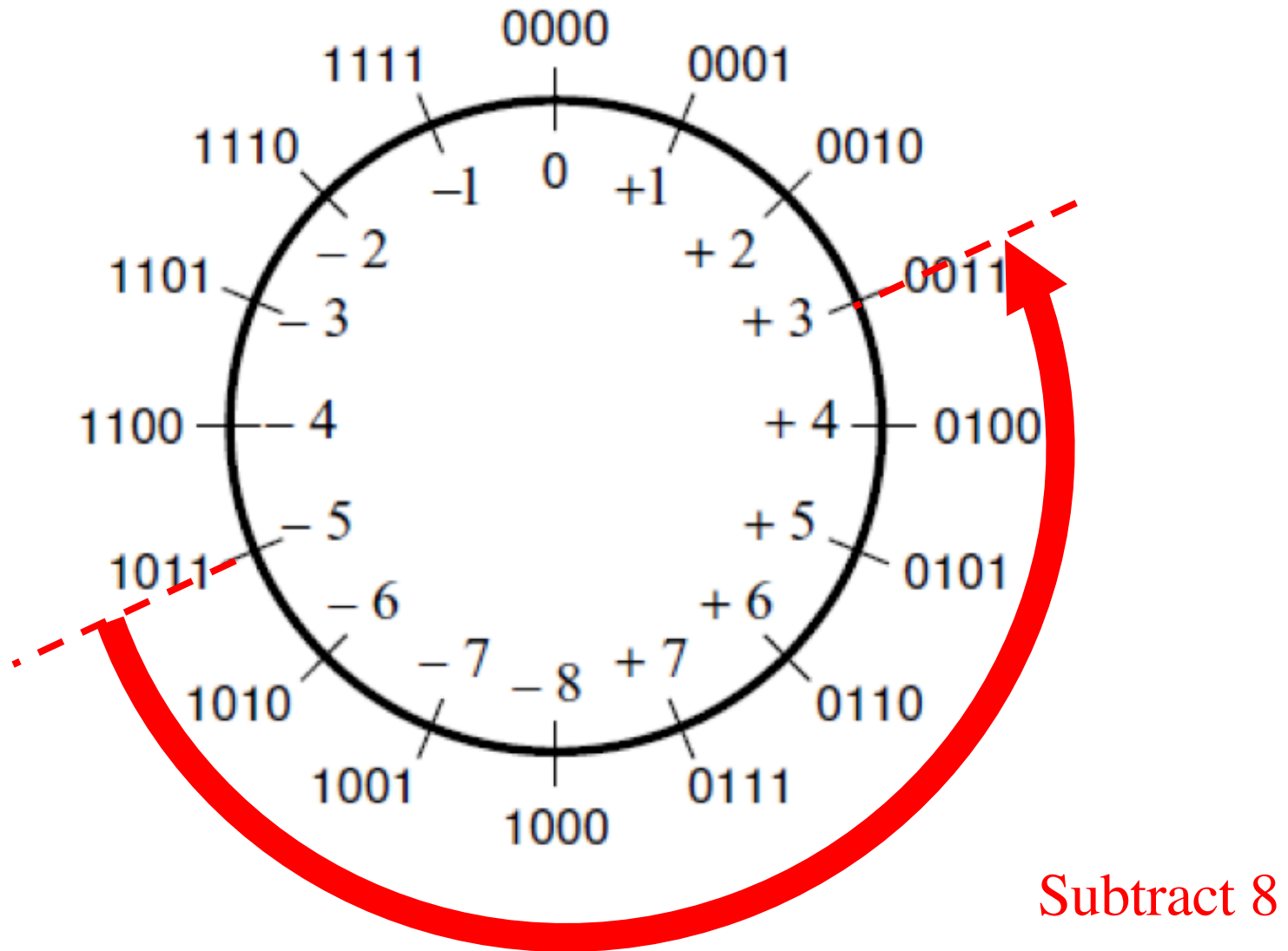
↑  
ignore

# But that exception does not matter

Add 8



# But that exception does not matter

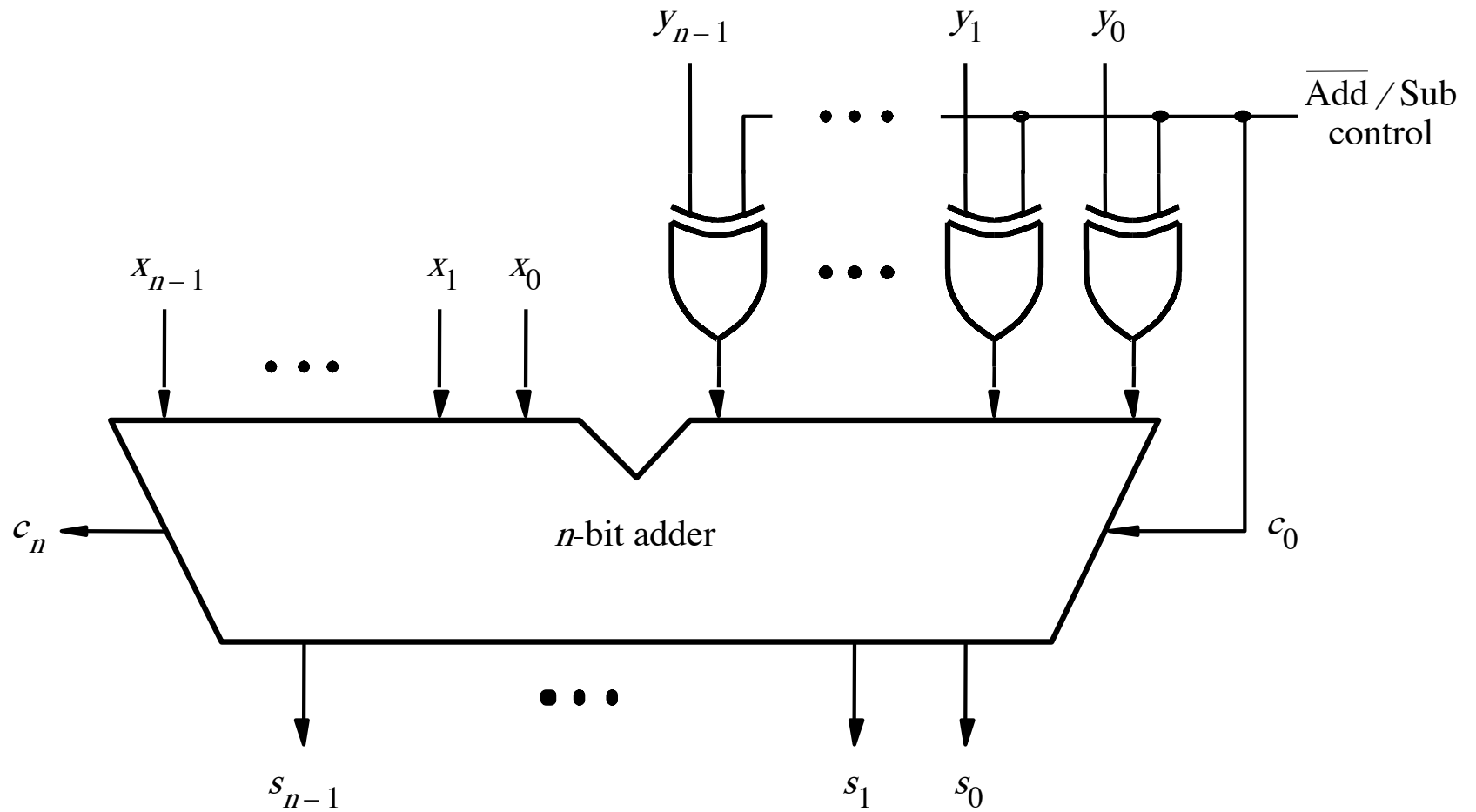


# Take-Home Message

- **Subtraction can be performed by simply adding the 2's complement of the second number, regardless of the signs of the two numbers.**
- **Thus, the same adder circuit can be used to perform both addition and subtraction !!!**



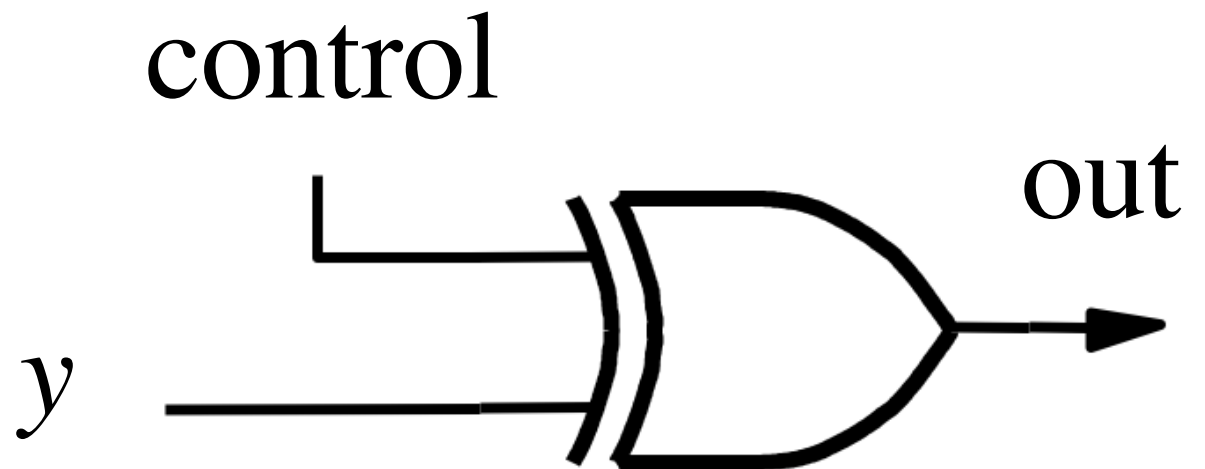
# Adder/subtractor unit



[ Figure 3.12 from the textbook ]

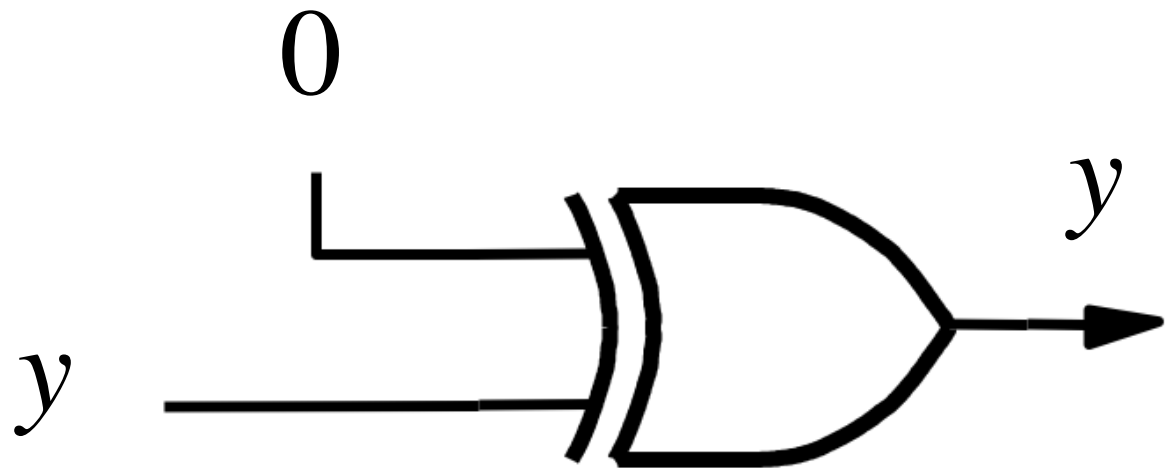

# XOR Tricks

control	$y$	out
0	0	0
0	1	1
1	0	1
1	1	0



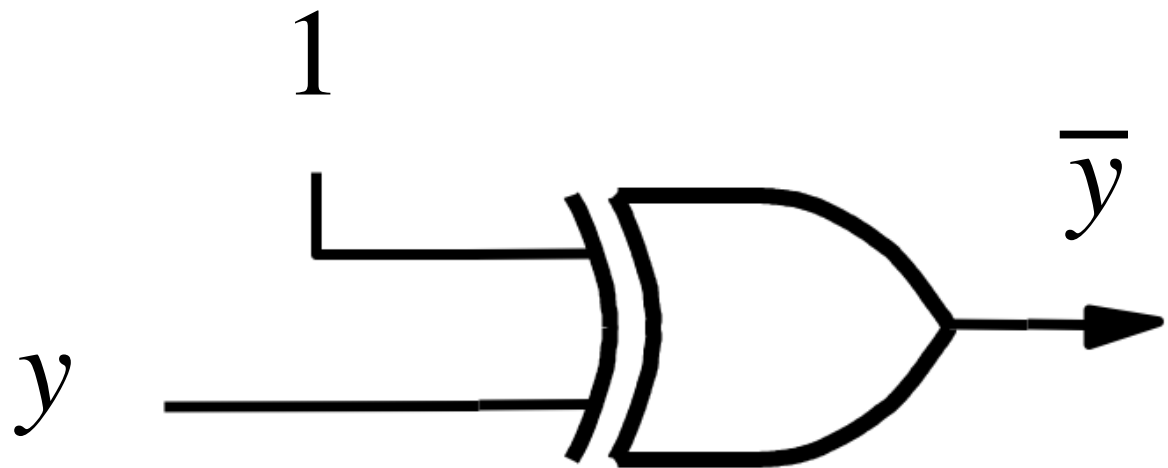
# XOR as a repeater

control	$y$	out
0	0	0
0	1	1

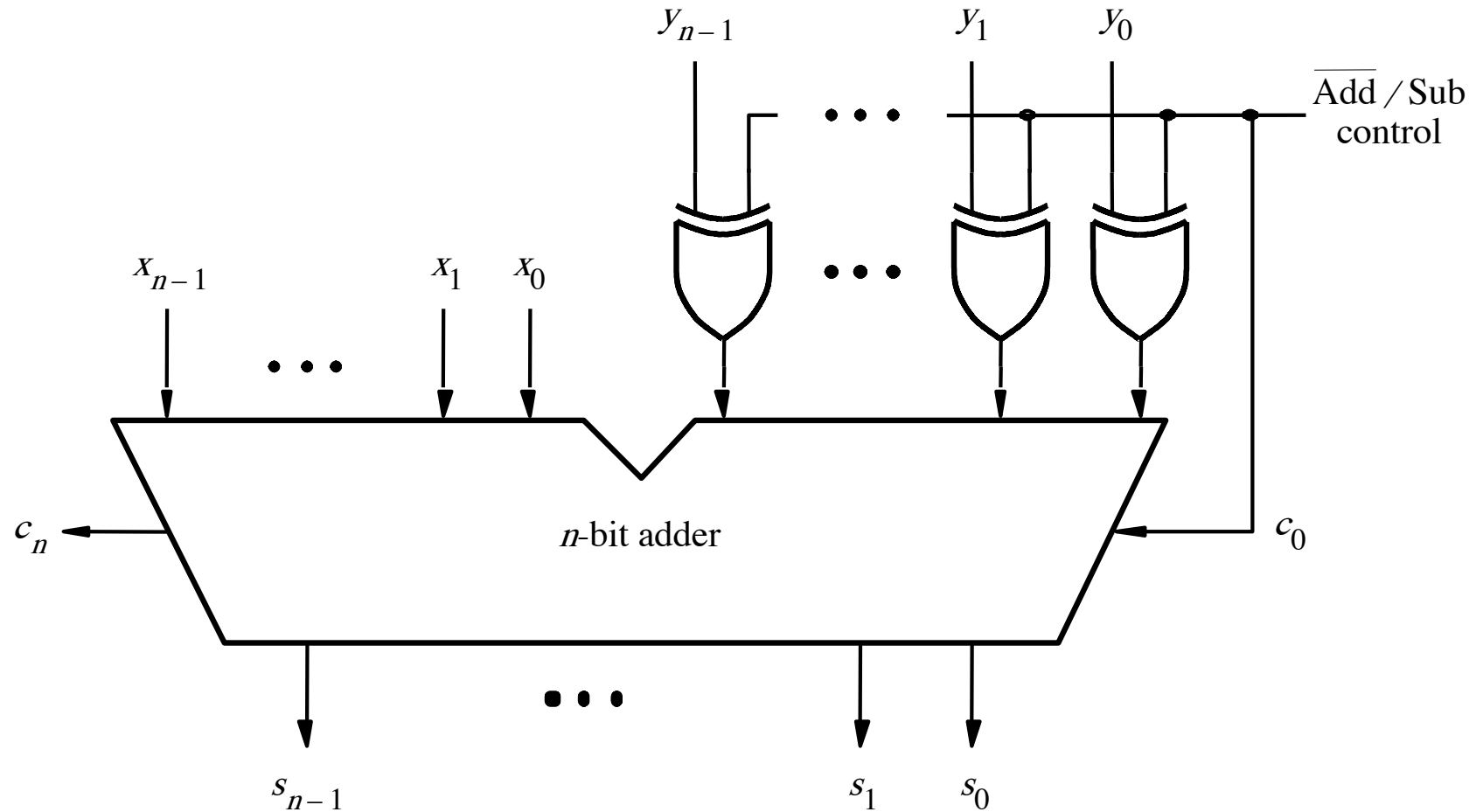


# XOR as an inverter

control	$y$	out
1	0	1
1	1	0

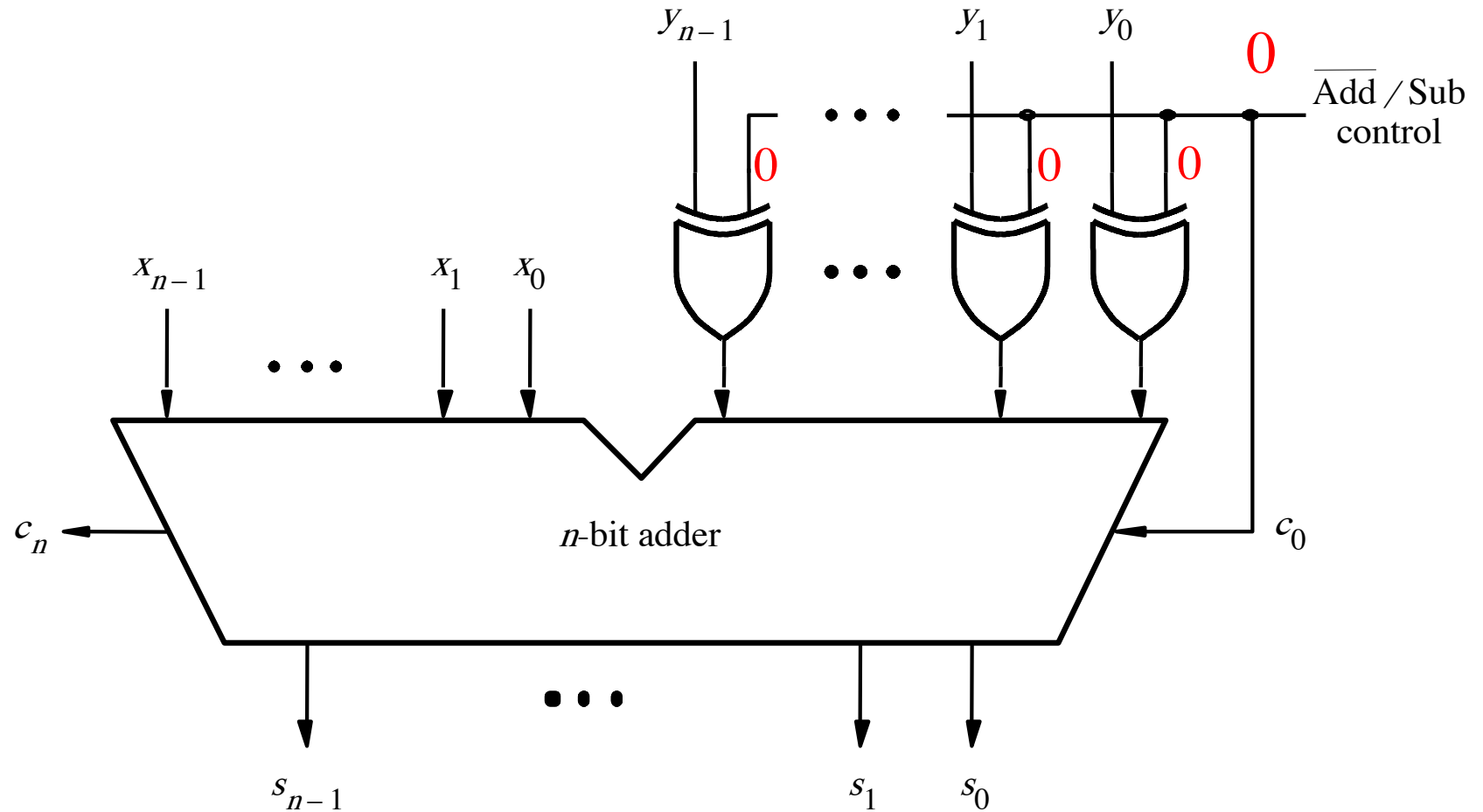


# Addition: when control = 0



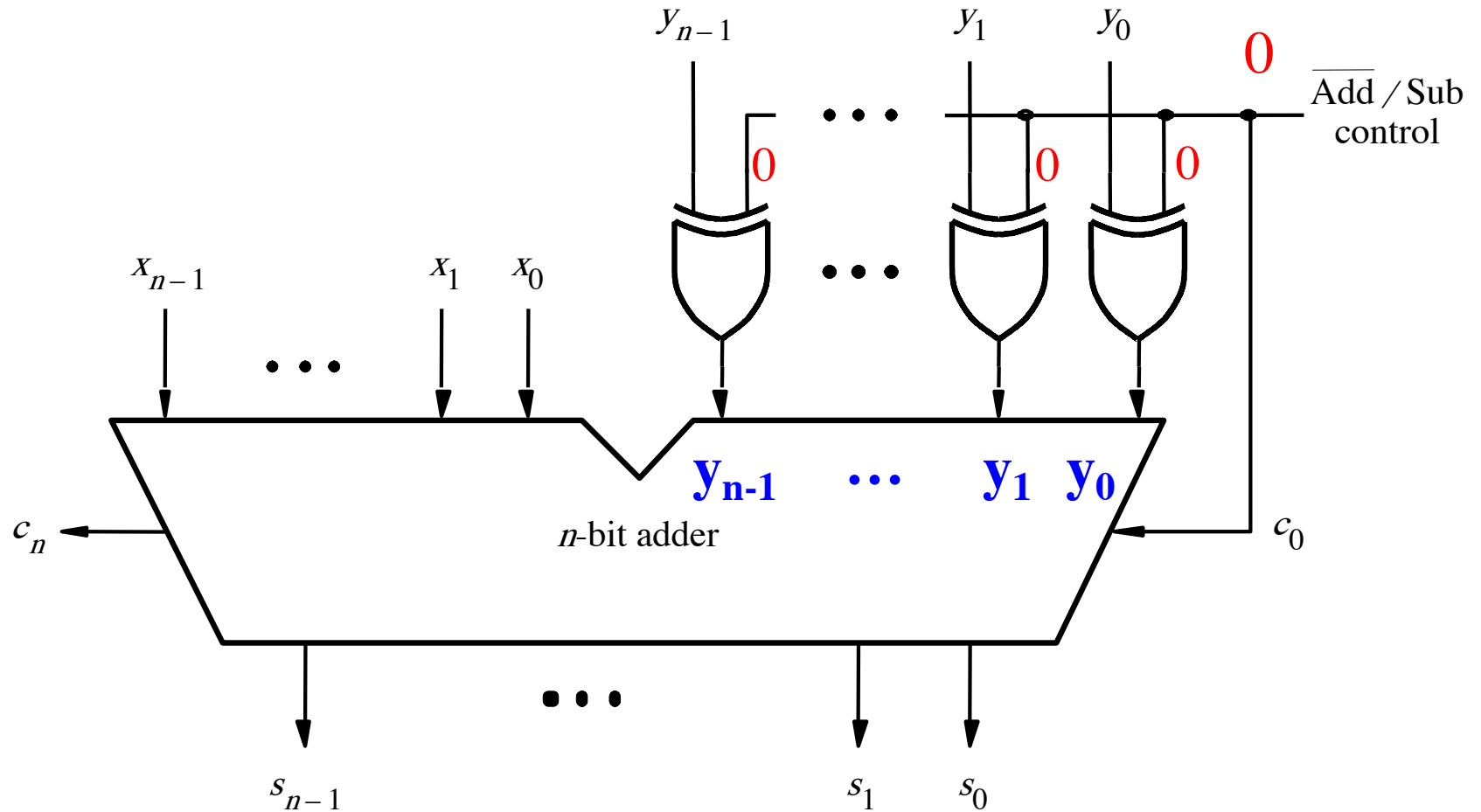
[ Figure 3.12 from the textbook ]

# Addition: when control = 0



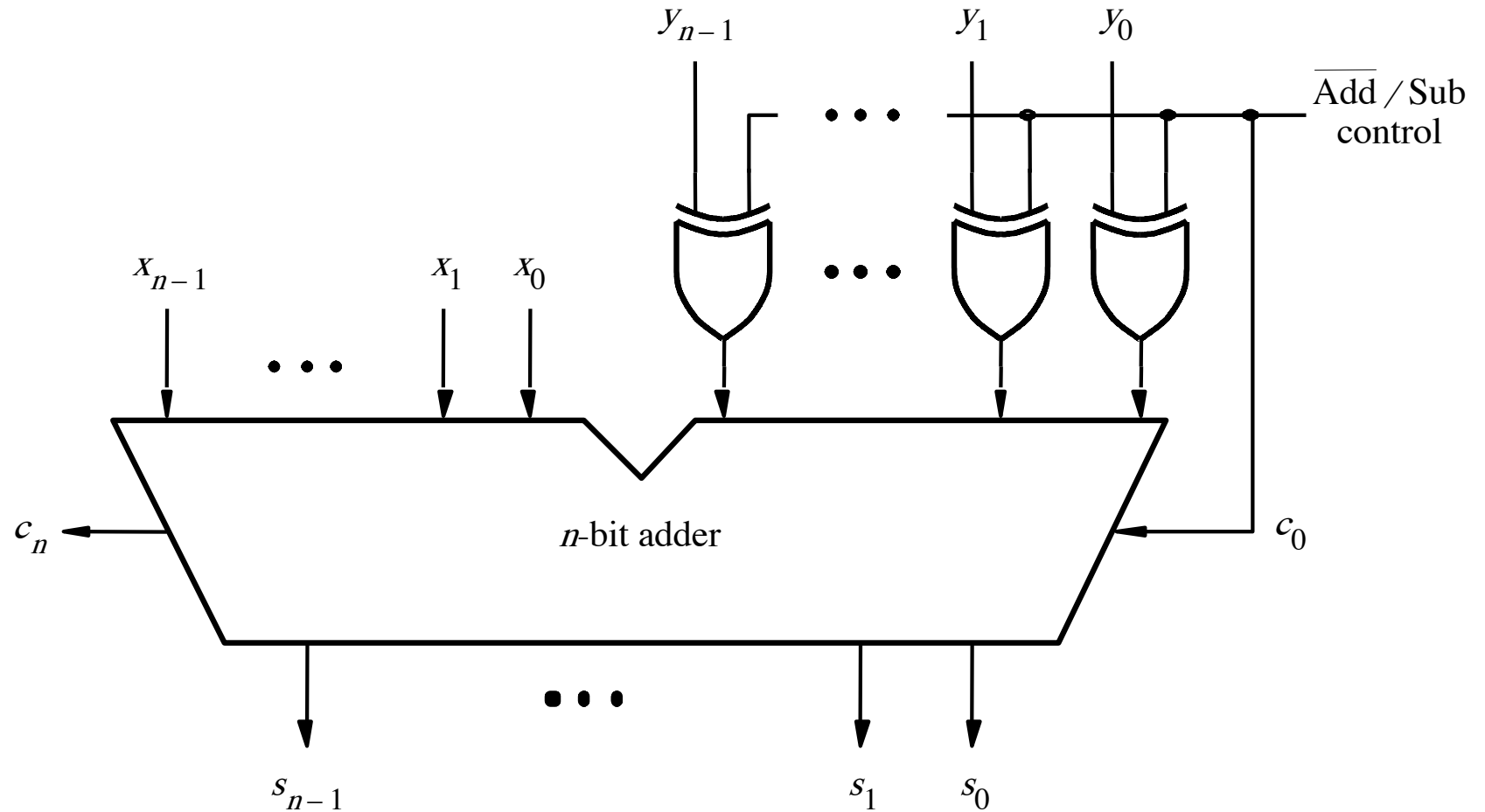
[ Figure 3.12 from the textbook ]

# Addition: when control = 0



[ Figure 3.12 from the textbook ]

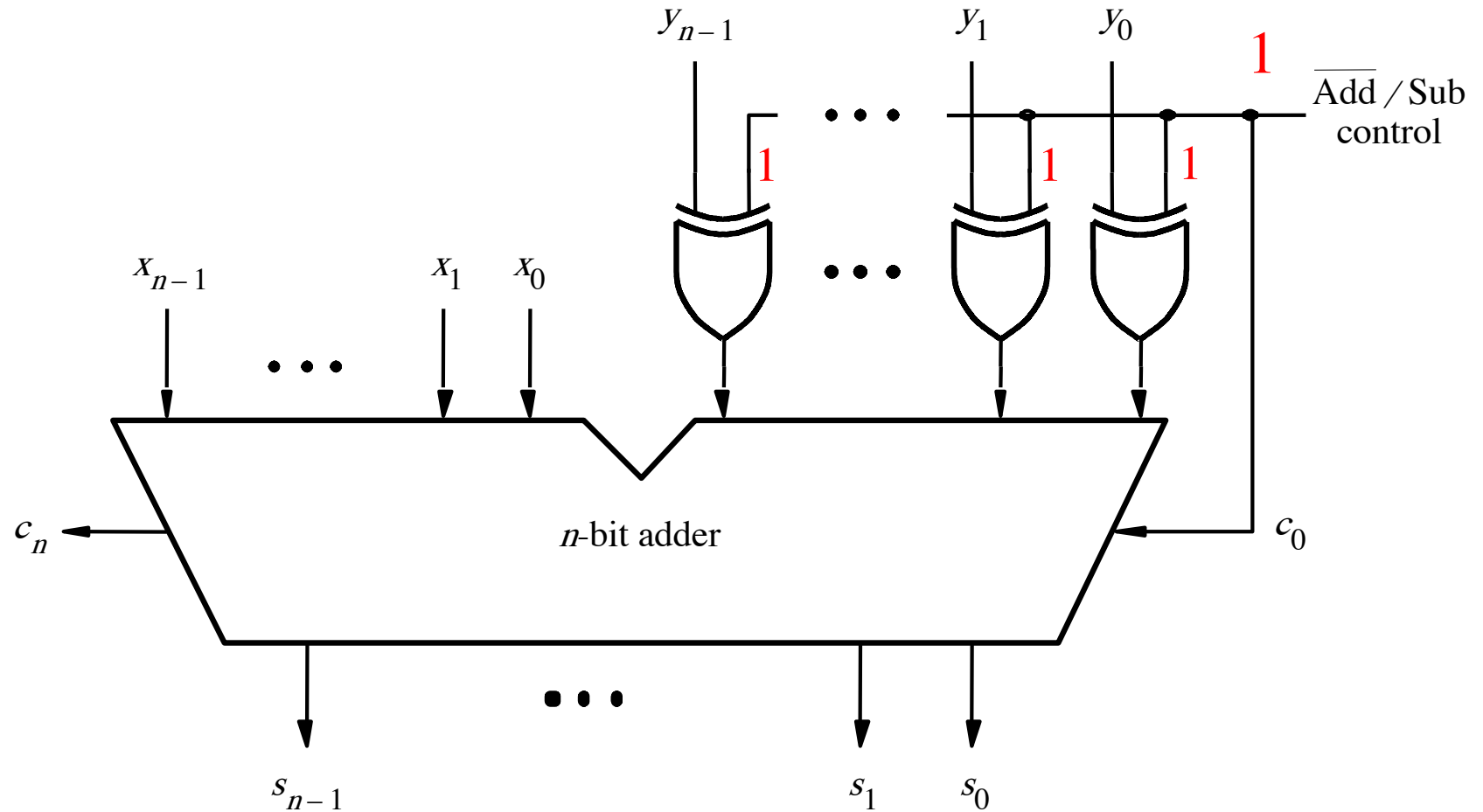
# Subtraction: when control = 1



[ Figure 3.12 from the textbook ]

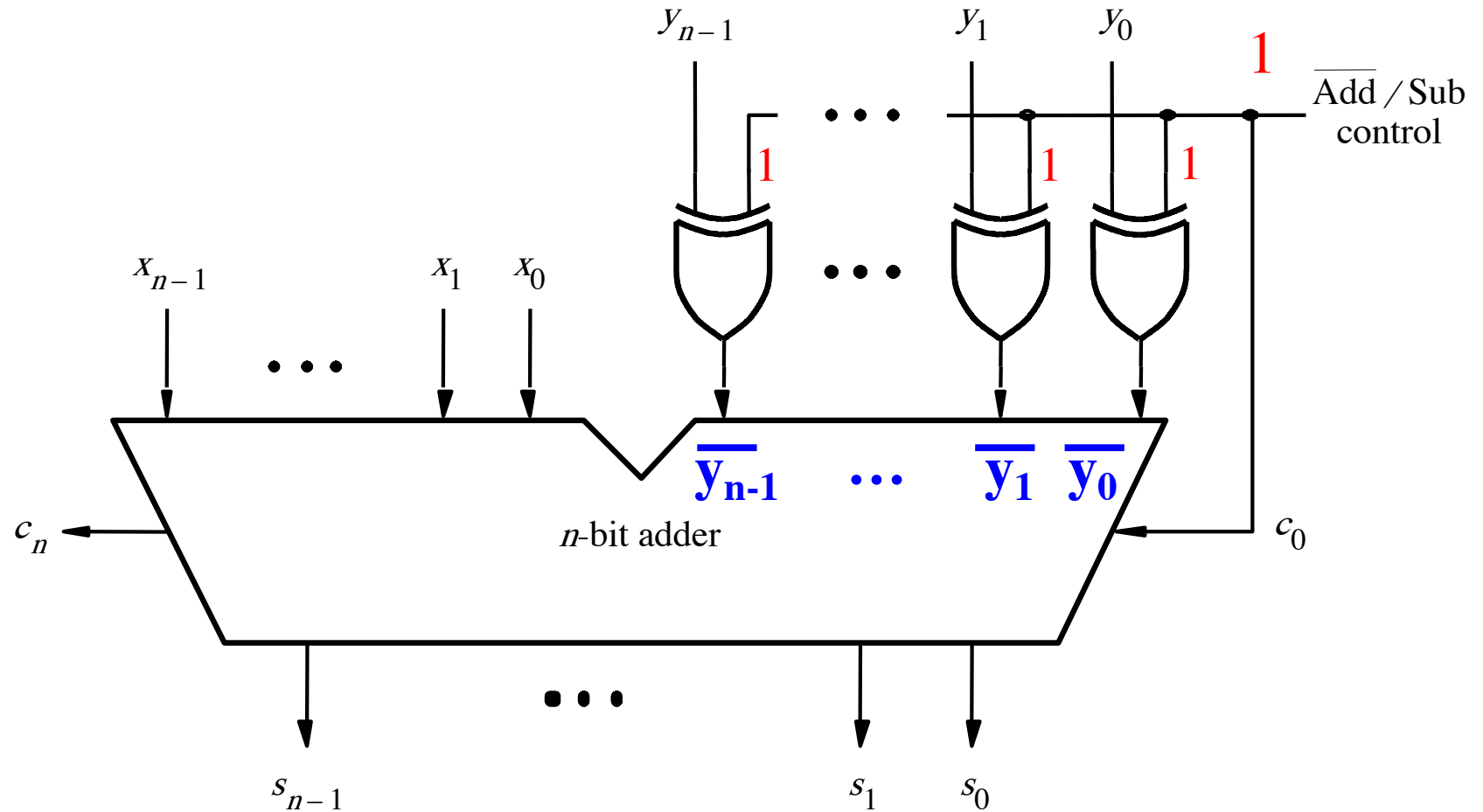


# Subtraction: when control = 1



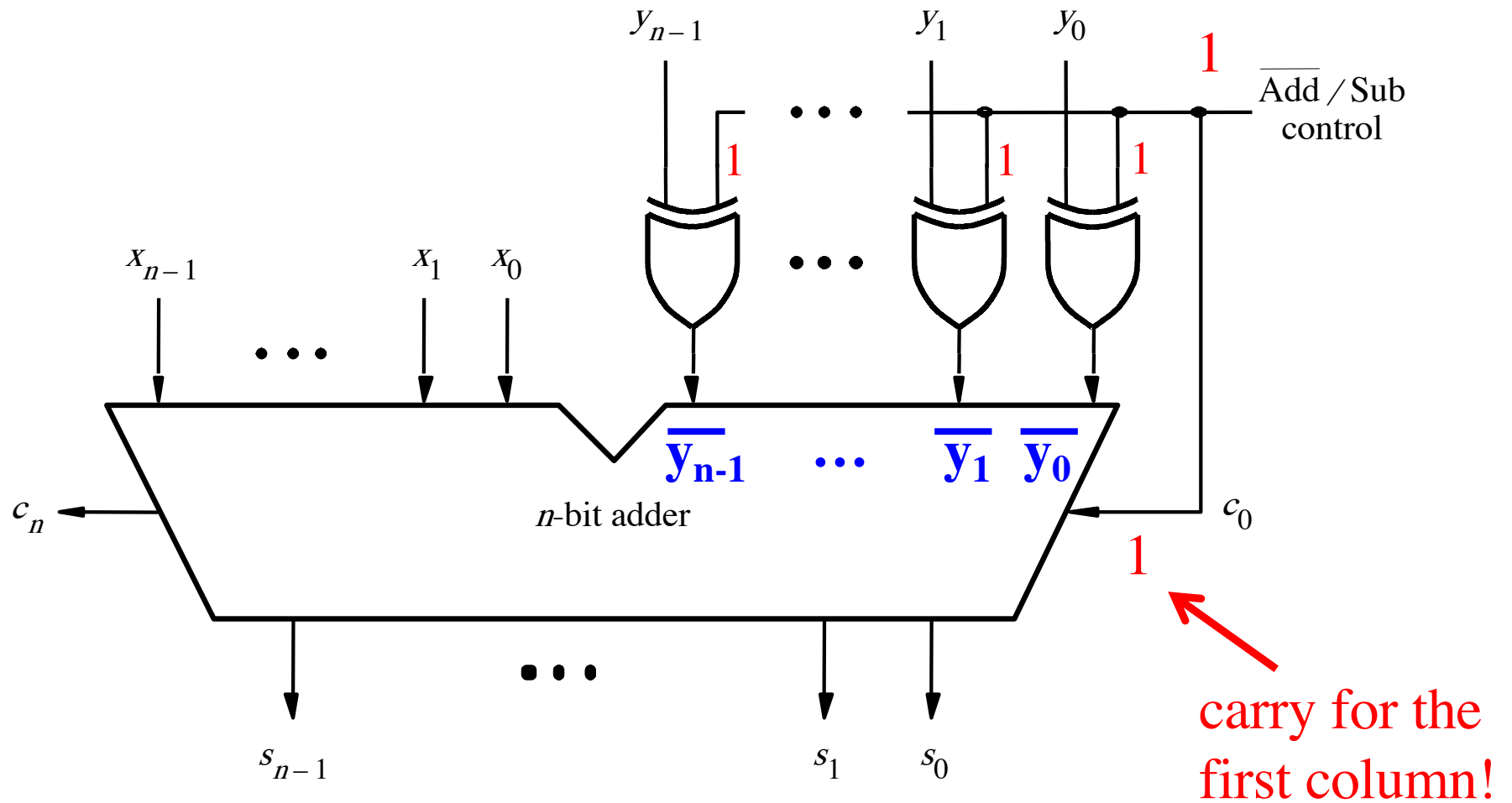
[ Figure 3.12 from the textbook ]

# Subtraction: when control = 1



[ Figure 3.12 from the textbook ]

# Subtraction: when control = 1



[ Figure 3.12 from the textbook ]

# Detecting Overflow

# Examples of determination of overflow

$$\begin{array}{r} (+7) \\ + (+2) \\ \hline (+9) \end{array} \quad + \quad \begin{array}{r} 0111 \\ 0010 \\ \hline 1001 \end{array}$$

$$\begin{array}{r} (-7) \\ + (+2) \\ \hline (-5) \end{array} \quad + \quad \begin{array}{r} 1001 \\ 0010 \\ \hline 1011 \end{array}$$

$$\begin{array}{r} (+7) \\ + (-2) \\ \hline (+5) \end{array} \quad + \quad \begin{array}{r} 0111 \\ 1110 \\ \hline 10101 \end{array}$$

$$\begin{array}{r} (-7) \\ + (-2) \\ \hline (-9) \end{array} \quad + \quad \begin{array}{r} 1001 \\ 1110 \\ \hline 10111 \end{array}$$

# Examples of determination of overflow

$$\begin{array}{r} (+7) \\ + (+2) \\ \hline (+9) \end{array} \quad \begin{array}{r} 01100 \\ + 0111 \\ \hline 0010 \\ + 1001 \end{array}$$

$$\begin{array}{r} (-7) \\ + (+2) \\ \hline (-5) \end{array} \quad \begin{array}{r} 00000 \\ + 1001 \\ \hline 0010 \\ + 1011 \end{array}$$

$$\begin{array}{r} (+7) \\ + (-2) \\ \hline (+5) \end{array} \quad \begin{array}{r} 11100 \\ + 0111 \\ \hline 1110 \\ + 10101 \end{array}$$

$$\begin{array}{r} (-7) \\ + (-2) \\ \hline (-9) \end{array} \quad \begin{array}{r} 10000 \\ + 1001 \\ \hline 1110 \\ + 10111 \end{array}$$

Include the carry bits:  $c_4 c_3 c_2 c_1 c_0$

# Examples of determination of overflow

$$\begin{array}{r}
 (+7) \\
 + (+2) \\
 \hline
 (+9)
 \end{array}
 \quad
 \begin{array}{r}
 \boxed{01}100 \\
 + \quad 0111 \\
 \quad 0010 \\
 \hline
 1001
 \end{array}$$

$$\begin{array}{r}
 (-7) \\
 + (+2) \\
 \hline
 (-5)
 \end{array}
 \quad
 \begin{array}{r}
 \boxed{00}000 \\
 + \quad 1001 \\
 \quad 0010 \\
 \hline
 1011
 \end{array}$$

$$\begin{array}{r}
 (+7) \\
 + (-2) \\
 \hline
 (+5)
 \end{array}
 \quad
 \begin{array}{r}
 \boxed{11}100 \\
 + \quad 0111 \\
 \quad 1110 \\
 \hline
 10101
 \end{array}$$

$$\begin{array}{r}
 (-7) \\
 + (-2) \\
 \hline
 (-9)
 \end{array}
 \quad
 \begin{array}{r}
 \boxed{10}000 \\
 + \quad 1001 \\
 \quad 1110 \\
 \hline
 10111
 \end{array}$$

Include the carry bits:  $\boxed{c_4 c_3} c_2 c_1 c_0$

# Examples of determination of overflow

$$c_4 = 0$$

$$c_3 = 1$$

$$\begin{array}{r}
 (+7) \\
 + (+2) \\
 \hline
 (+9)
 \end{array}
 \quad
 \begin{array}{r}
 \boxed{01}100 \\
 + \quad 0111 \\
 \hline
 \quad 1001
 \end{array}$$

$$\begin{array}{r}
 (-7) \\
 + (+2) \\
 \hline
 (-5)
 \end{array}
 \quad
 \begin{array}{r}
 \boxed{00}000 \\
 + \quad 1001 \\
 \hline
 \quad 1011
 \end{array}
 \quad
 \begin{array}{l}
 c_4 = 0 \\
 c_3 = 0
 \end{array}$$

$$c_4 = 1$$

$$c_3 = 1$$

$$\begin{array}{r}
 (+7) \\
 + (-2) \\
 \hline
 (+5)
 \end{array}
 \quad
 \begin{array}{r}
 \boxed{11}100 \\
 + \quad 0111 \\
 \hline
 \quad 1110 \\
 \hline
 10101
 \end{array}$$

$$\begin{array}{r}
 (-7) \\
 + (-2) \\
 \hline
 (-9)
 \end{array}
 \quad
 \begin{array}{r}
 \boxed{10}000 \\
 + \quad 1001 \\
 \hline
 \quad 1110 \\
 \hline
 10111
 \end{array}
 \quad
 \begin{array}{l}
 c_4 = 1 \\
 c_3 = 0
 \end{array}$$

Include the carry bits:  $\boxed{c_4 c_3} c_2 c_1 c_0$



# Examples of determination of overflow

$$c_4 = 0$$

$$c_3 = 1$$

$$\begin{array}{r} (+7) \\ + (+2) \\ \hline (+9) \end{array}$$

$$\begin{array}{r} \boxed{01}100 \\ + 0111 \\ \hline 0010 \\ \hline 1001 \end{array}$$

$$\begin{array}{r} (-7) \\ + (+2) \\ \hline (-5) \end{array}$$

$$\begin{array}{r} \boxed{00}000 \\ + 1001 \\ \hline 0010 \\ \hline 1011 \end{array}$$

$$c_4 = 0$$

$$c_3 = 0$$

$$c_4 = 1$$

$$c_3 = 1$$

$$\begin{array}{r} (+7) \\ + (-2) \\ \hline (+5) \end{array}$$

$$\begin{array}{r} \boxed{11}100 \\ + 0111 \\ \hline 1110 \\ \hline 10101 \end{array}$$

$$\begin{array}{r} (-7) \\ + (-2) \\ \hline (-9) \end{array}$$

$$\begin{array}{r} \boxed{10}000 \\ + 1001 \\ \hline 1110 \\ \hline 10111 \end{array}$$

$$c_4 = 1$$

$$c_3 = 0$$

Overflow occurs only in these two cases.

# Examples of determination of overflow

$$c_4 = 0$$

$$c_3 = 1$$

$$\begin{array}{r} (+7) \\ + (+2) \\ \hline (+9) \end{array}$$

$$\begin{array}{r} \boxed{01}100 \\ + \quad 0111 \\ \hline 0010 \\ \hline 1001 \end{array}$$

$$\begin{array}{r} (-7) \\ + (+2) \\ \hline (-5) \end{array}$$

$$\begin{array}{r} \boxed{00}000 \\ + \quad 1001 \\ \hline 0010 \\ \hline 1011 \end{array}$$

$$c_4 = 0$$

$$c_3 = 0$$

$$c_4 = 1$$

$$c_3 = 1$$

$$\begin{array}{r} (+7) \\ + (-2) \\ \hline (+5) \end{array}$$

$$\begin{array}{r} \boxed{11}100 \\ + \quad 0111 \\ \hline 1110 \\ \hline 10101 \end{array}$$

$$\begin{array}{r} (-7) \\ + (-2) \\ \hline (-9) \end{array}$$

$$\begin{array}{r} \boxed{10}000 \\ + \quad 1001 \\ \hline 1110 \\ \hline 10111 \end{array}$$

$$c_4 = 1$$

$$c_3 = 0$$

$$\text{Overflow} = c_3 \bar{c}_4 + \bar{c}_3 c_4$$

# Examples of determination of overflow

$$c_4 = 0$$

$$c_3 = 1$$

$$\begin{array}{r} (+7) \\ + (+2) \\ \hline (+9) \end{array}$$

$$\begin{array}{r} \boxed{01}100 \\ + 0111 \\ \hline 0010 \\ \hline 1001 \end{array}$$

$$\begin{array}{r} (-7) \\ + (+2) \\ \hline (-5) \end{array}$$

$$\begin{array}{r} \boxed{00}000 \\ + 1001 \\ \hline 0010 \\ \hline 1011 \end{array}$$

$$c_4 = 0$$

$$c_3 = 0$$

$$c_4 = 1$$

$$c_3 = 1$$

$$\begin{array}{r} (+7) \\ + (-2) \\ \hline (+5) \end{array}$$

$$\begin{array}{r} \boxed{11}100 \\ + 0111 \\ \hline 1110 \\ \hline 10101 \end{array}$$

$$\begin{array}{r} (-7) \\ + (-2) \\ \hline (-9) \end{array}$$

$$\begin{array}{r} \boxed{10}000 \\ + 1001 \\ \hline 1110 \\ \hline 10111 \end{array}$$

$$c_4 = 1$$

$$c_3 = 0$$

$$\text{Overflow} = \underbrace{c_3\bar{c}_4 + \bar{c}_3c_4}_{\text{XOR}}$$

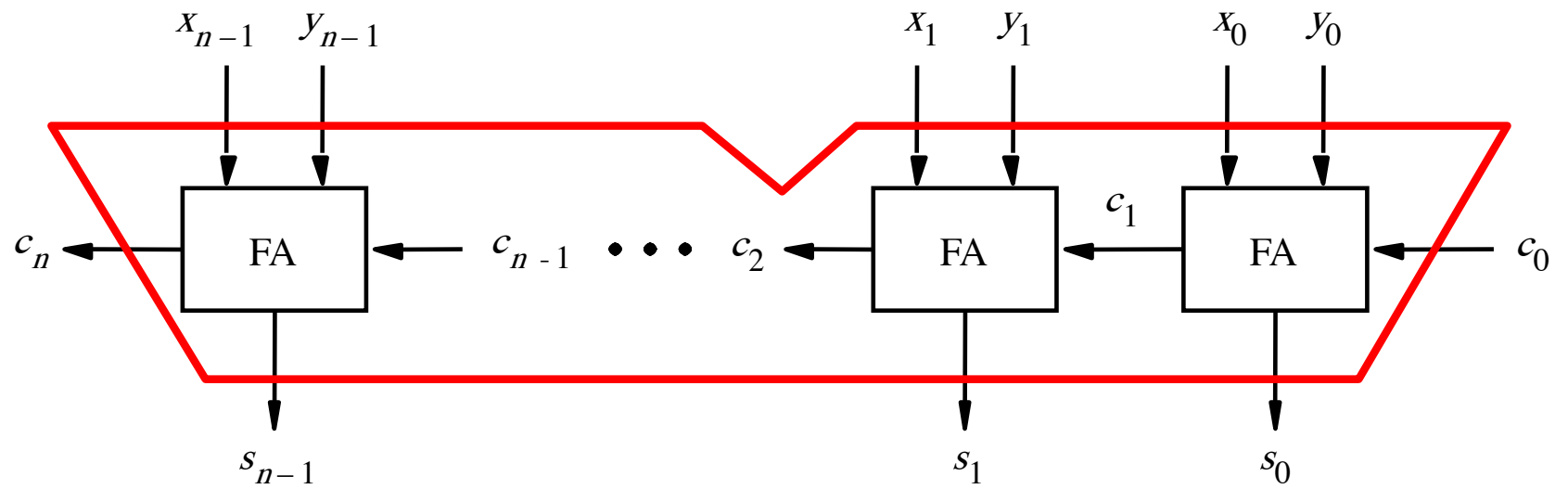
# Calculating overflow for 4-bit numbers with only three significant bits

$$\begin{aligned}\text{Overflow} &= c_3 \bar{c}_4 + \bar{c}_3 c_4 \\ &= c_3 \oplus c_4\end{aligned}$$

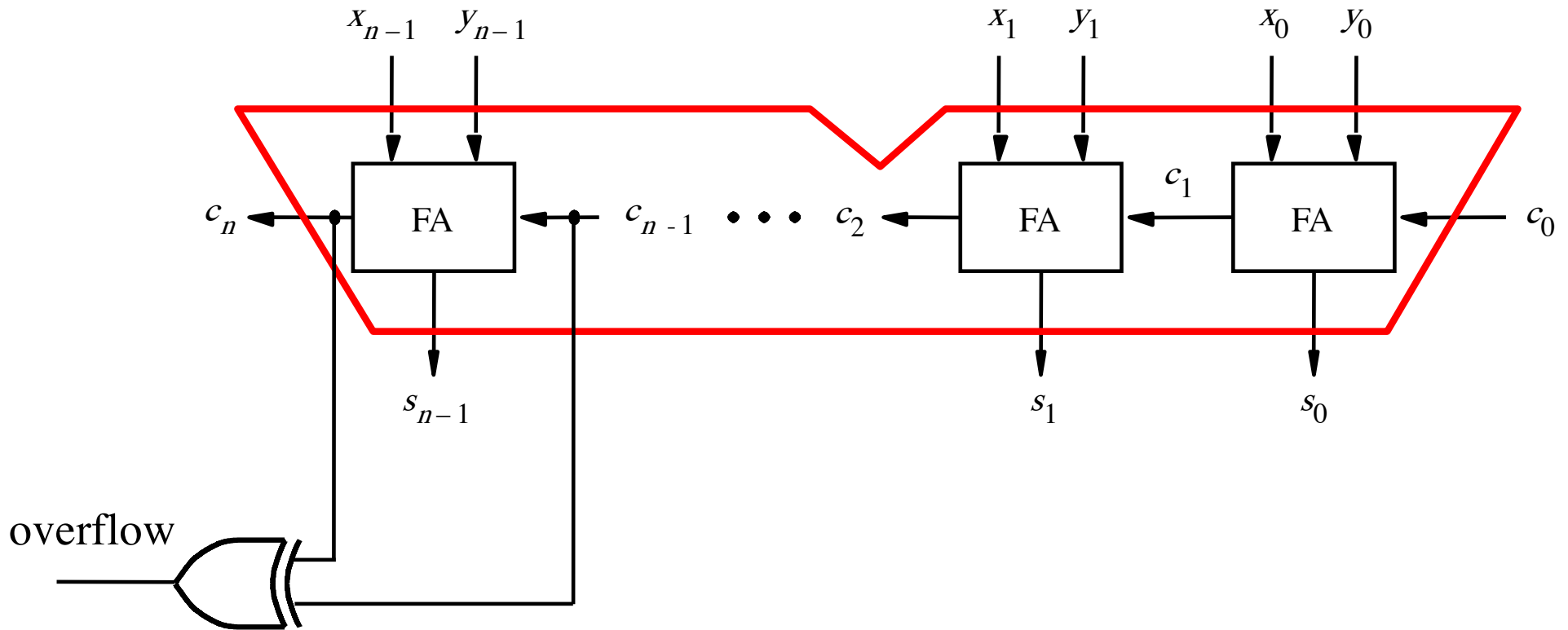
# Calculating overflow for n-bit numbers with only n-1 significant bits

$$\text{Overflow} = c_{n-1} \oplus c_n$$

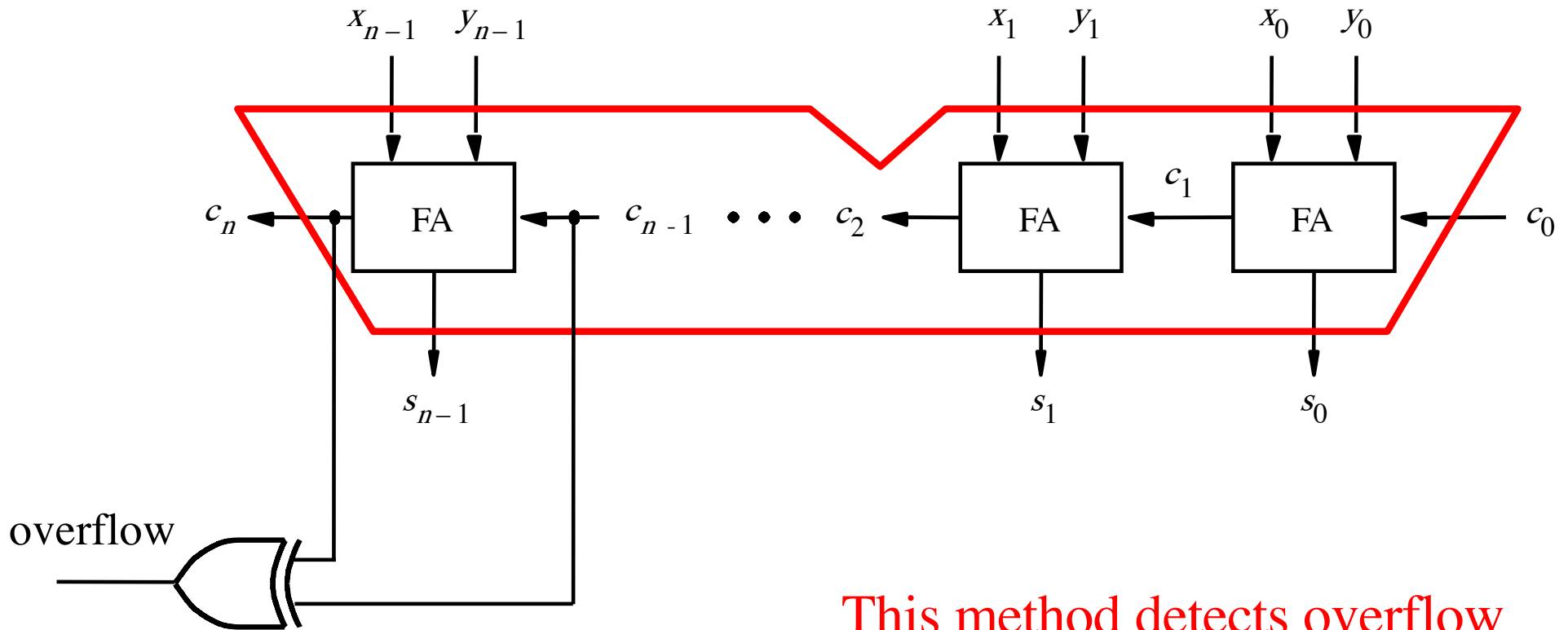
# Detecting Overflow



# Detecting Overflow (with one extra XOR)



# Detecting Overflow (with one extra XOR)



This method detects overflow  
for both addition and subtraction.



# Detecting Overflow

(alternative method that **works only for addition**)

# Another way to look at the overflow issue

$$\begin{array}{r} + \quad X = \quad x_3 \quad x_2 \quad x_1 \quad x_0 \\ \quad Y = \quad y_3 \quad y_2 \quad y_1 \quad y_0 \\ \hline \quad S = \quad s_3 \quad s_2 \quad s_1 \quad s_0 \end{array}$$

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$$\begin{array}{rcccc} + & X = & x_3 & x_2 & x_1 & x_0 \\ & Y = & y_3 & y_2 & y_1 & y_0 \\ \hline & S = & s_3 & s_2 & s_1 & s_0 \end{array}$$

If both numbers that we are adding have the same sign but the sum does not, then we have an overflow.

# Examples of determination of overflow

$$\begin{array}{r} (+7) \\ + (+2) \\ \hline (+9) \end{array} \quad + \quad \begin{array}{r} 0111 \\ 0010 \\ \hline 1001 \end{array}$$

$$\begin{array}{r} (-7) \\ + (+2) \\ \hline (-5) \end{array} \quad + \quad \begin{array}{r} 1001 \\ 0010 \\ \hline 1011 \end{array}$$

$$\begin{array}{r} (+7) \\ + (-2) \\ \hline (+5) \end{array} \quad + \quad \begin{array}{r} 0111 \\ 1110 \\ \hline 10101 \end{array}$$

$$\begin{array}{r} (-7) \\ + (-2) \\ \hline (-9) \end{array} \quad + \quad \begin{array}{r} 1001 \\ 1110 \\ \hline 10111 \end{array}$$

# Examples of determination of overflow

$$\begin{array}{r} (+7) \\ + (+2) \\ \hline (+9) \end{array} \quad + \quad \begin{array}{r} \boxed{0}111 \\ 0010 \\ \hline \boxed{1}001 \end{array}$$

$$\begin{array}{r} (-7) \\ + (+2) \\ \hline (-5) \end{array} \quad + \quad \begin{array}{r} \boxed{1}001 \\ 0010 \\ \hline \boxed{1}011 \end{array}$$

$$\begin{array}{r} (+7) \\ + (-2) \\ \hline (+5) \end{array} \quad + \quad \begin{array}{r} \boxed{0}111 \\ 1110 \\ \hline 1\boxed{0}101 \end{array}$$

$$\begin{array}{r} (-7) \\ + (-2) \\ \hline (-9) \end{array} \quad + \quad \begin{array}{r} \boxed{1}001 \\ 1110 \\ \hline 1\boxed{0}111 \end{array}$$

# Examples of determination of overflow

$$\begin{aligned}x_3 &= 0 \\y_3 &= 0 \\s_3 &= 1\end{aligned}$$

$$\begin{array}{r} (+7) \\ + (+2) \\ \hline (+9) \end{array} \quad + \quad \begin{array}{r} \boxed{0}111 \\ \boxed{0}010 \\ \hline \boxed{1}001 \end{array}$$

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$$\begin{aligned}x_3 &= 1 \\y_3 &= 0 \\s_3 &= 1\end{aligned}$$

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$$\begin{array}{r} (+7) \\ + (-2) \\ \hline (+5) \end{array} \quad + \quad \begin{array}{r} \boxed{0}111 \\ \boxed{1}110 \\ \hline 1\boxed{0}101 \end{array}$$

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$$\begin{array}{r} \boxed{1}001 \\ \boxed{1}110 \\ \hline 1\boxed{0}111 \end{array}$$

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$$\begin{aligned} x_3 &= 1 \\ y_3 &= 1 \\ s_3 &= 0 \end{aligned}$$

In 2's complement, both +9 and -9 are not representable with 4 bits.

# Examples of determination of overflow

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$$\begin{aligned} x_3 &= 1 \\ y_3 &= 1 \\ s_3 &= 0 \end{aligned}$$

Overflow occurs only in these two cases.



# Examples of determination of overflow

$$\begin{aligned} x_3 &= 0 \\ y_3 &= 0 \\ s_3 &= 1 \end{aligned}$$

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$$\begin{aligned} x_3 &= 1 \\ y_3 &= 0 \\ s_3 &= 1 \end{aligned}$$

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$$+ \begin{array}{r} 1001 \\ 1110 \\ \hline 10111 \end{array}$$

$$\begin{aligned} x_3 &= 1 \\ y_3 &= 1 \\ s_3 &= 0 \end{aligned}$$

$$\text{Overflow} = \bar{x}_3 \bar{y}_3 s_3 + x_3 y_3 \bar{s}_3$$

# Another way to look at the overflow issue

$$\begin{array}{rcccc} + & X = & x_3 & x_2 & x_1 & x_0 \\ & Y = & y_3 & y_2 & y_1 & y_0 \\ \hline & S = & s_3 & s_2 & s_1 & s_0 \end{array}$$

If both numbers that we are adding have the same sign but the sum does not, then we have an overflow.

$$\text{Overflow} = \bar{x}_3 \bar{y}_3 s_3 + x_3 y_3 \bar{s}_3$$

**Questions?**

**THE END**